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A. E. Green and R. T. Shield

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FINITE EXTENSION AND TORSION OF CYLINDERS

By A. E. GREEN and R. T. SHIELD
Mathematics Department, King's College, Newcastle upon Tyne

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CONTENTS

The theory of finite elastic deformations of an isotropic body, in which a completely general strain-energy function is used, is applied to the problem of a small twist superposed upon a finite extension of a cylinder which has a constant cross-section. The law which relates the force necessary to produce the large extension, with the torsional modulus for the small torsion superposed on that extension, is given by a simple general formula. When the material is incompressible the corresponding law is independent of the particular form of the strain-energy function which applies to the material. When the cylinder is not a circular cylinder or a circular cylindrical tube the twisting couple vanishes for a certain value of the extension ratio, this value being independent of the particular form of the strain-energy function when the material is incompressible. The problems of a small twist superposed upon a hydrostatic pressure, or upon a combined hydrostatic pressure and tension, are also solved.

Attention is then confined to isotropic incompressible rubber-like materials using a strain-energy function suggested by Mooney, and the second-order effects which accompany the torsion of cylinders of constant cross-sections are examined. The problem is reduced to the determination of two functions of a complex variable which are regular in the cross-section of the cylinders and which satisfy a suitable boundary condition on the boundary of the cross-section. The solution is expressed as an integral equation and applications are made to cylinders with various cross-sections. This theory is then generalized to include the second-order effects in torsion superposed upon a finite extension of the cylinders. Complex variables are used throughout this part of the paper, and the problem is reduced to the determination of four canonical functions of a complex variable, these functions being the solutions of certain integral equations. An explicit solution is given for an elliptical cylinder but without using the integral equations.

1. INTRODUCTION

The theory of finite elastic deformation of incompressible isotropic bodies has been considered by Rivlin (see references) and more recently by Green & Shield (1950), and special problems have been solved using completely general forms for the strain-energy function. Oldroyd (1950) has also discussed the problem of the torsion of an anisotropic circular cylinder of (almost) incompressible material. It has been possible to obtain exact solutions of the fundamental equations in the problems which have been considered so far because the displacements had certain symmetrical properties. For more general types of problems, however, it does not appear to be possible to obtain exact solutions owing to the non-linearity of the differential equations occurring in the theory, and it is natural to consider approximations to exact solutions.

By assuming a special stress-strain relationship Seth and Shepherd (see references) have solved a number of problems for compressible materials, but Rivlin (1948*d*) has pointed out that Seth's stress-strain relations are not allowable for the description of the elastic properties of an isotropic material in which the strain-energy is a function of the strain invariants only. A number of other writers (see references) have determined second-order effects in a variety of problems, some of these writers using a stress-strain relationship given by Murnaghan (1937) for isotropic compressible bodies.

Second-order effects have also been considered by Biot (1939*b*, *c*, 1940) and Goodier (1950) using a somewhat different approach based on theory developed by Biot (1939*a*). Biot derives a non-linear theory of elasticity and then considers a linearized case for a body which is under initial stress. Goodier examines the problem of the small twist of a cylinder which is subjected to a non-uniform initial stress.

In the present paper we first reconsider the problem of a small twist superposed upon a uniform finite extension of an isotropic cylinder which has a constant cross-section. The complete deformation is examined from an initial state in which the cylinder is unstrained and unstressed, and the problem is solved for a compressible material using a completely general strain-energy function. The law which relates the force necessary to produce the large simple extension, with the torsional modulus for a small torsion superposed on that simple extension, is given by a general formula (3·29). When the material is incompressible the corresponding law (3·30) is independent of the particular form of the strain-energy function which applies to the material. This law is a generalization of a result found by Rivlin (1949*c*) for a cylinder with a circular cross-section which has already been verified experimentally by Rivlin & Saunders (1951) for rubber-like materials. When the cylinder is not a circular cylinder or a circular cylindrical tube the twisting couple vanishes when the extension ratios are related by equation (3·27). When the cylinder is incompressible the twisting couple vanishes when the extension ratio λ along the length of the cylinder is given by (3·28), and this value of λ is independent of the particular form of the strain-energy function which applies to the material. In § 4 the solutions of the problems of a small twist of a cylinder superposed upon a hydrostatic pressure, or upon a combined hydrostatic pressure and tension, are also given, again using a completely general form for the strain-energy function. The formulae for the twisting couple for the latter problem are given by (4·16) for a compressible material, and by (4·18) for an incompressible material.

In the rest of the paper attention is confined to incompressible materials, and a special strain-energy function suggested by Mooney (1940) for rubber-like materials is used. We consider the secondary effects accompanying the torsion of cylinders with constant cross-sections. This problem has been considered by Ishlinsky (1943), Panov (1939), Riz (1938, 1943), Riz & Zvolinsky (1938) and Zvolinsky (1939) using Murnaghan's stress-strain relationship for compressible bodies, but this work is not very suitable for rubber-like materials. Also the methods used here, and the mathematical form of the results, differ from those of previous writers, but the general characteristics of the deformation are similar. The problem is formulated in § 5 in terms of co-ordinates in the unstrained body and is then reduced in § 7 to the determination of two functions of a complex variable which are regular in the cross-section of the cylinder and which satisfy a boundary condition on the boundary of the cross-section. When the mapping function which maps the cross-section upon the interior of the unit circle is known, the displacements and stresses can be expressed in terms of the solution of a certain integral equation (7·18). It is found that there is no additional warping of the cross-section beyond that given by the infinitesimal theory, and if the cylinder is not to increase in length the torsion can only be maintained by an additional force applied at one end of the cylinder parallel to its axis. Alternatively, the additional force can be made to vanish if the cylinder is allowed to change in length. The theory is illustrated in §§ 8, 9 by applications to cylinders whose cross-sections are a cardioid, an epitrochoid and a Booth's lemniscate.

In the final sections of the paper the theory just described is generalized by considering second-order effects in torsion after the cylinder has received a finite uniform extension along its length, the axis of torsion, however, being the line of centroids of cross-sections. Here complex variable notations are used at the outset, and the solution is reduced to the determination of four canonical functions of a complex variable which satisfy certain integral equations (7·18) and (10·24) so that applications may be made once the mapping function of the cross-section is known. An explicit solution is given in § 11 for an elliptical cylinder without using the integral equations.

2. Notation and formulae

We use the theory developed in two recent papers (Green & Zerna 1950; Green & Shield 1950), which we shall refer to as I, II respectively, and for conciseness we shall repeat only the formulae required in the following work.

As in I, the points P_0 of an unstrained and unstressed elastic body B_0 at rest are defined at time $t = 0$ by a rectangular Cartesian system of co-ordinates x_i or by a general curvilinear system of co-ordinates θ_i, where

$$x_i = x_i(\theta_1, \theta_2, \theta_3), \tag{2·1}$$

and the line element ds_0 of B_0 is given by

$$ds_0^2 = dx^i\,dx^i = g_{ik}\,d\theta^i\,d\theta^k. \tag{2·2}$$

The usual summation convention is used and Latin indices take the values 1, 2, 3. An index which is repeated more than twice is not summed.

The body B_0 is now strained or deformed, so that at time t the points P_0 of B_0 have moved to new positions P to form a strained body B. The points P of B are defined by a new rectangular Cartesian system of co-ordinates y_i which are related to x_i by the equations

$$y_i = f_i(x_1, x_2, x_3, t). \tag{2·3}$$

The curvilinear co-ordinates θ_i in B_0, which move with the body as it is deformed, form a curvilinear system θ_i in B so that

$$y_i = y_i(\theta_1, \theta_2, \theta_3, t), \tag{2·4}$$

and the line element ds in B, for a given time t, is given by

$$ds^2 = dy^i\,dy^i = G_{ik}\,d\theta^i\,d\theta^k. \tag{2·5}$$

We define covariant components of a strain tensor as

$$\gamma_{ik} = \tfrac{1}{2}(G_{ik} - g_{ik}). \tag{2·6}$$

In the curvilinear system θ_i covariant and contravariant base vectors $\mathbf{e}_i$, $\mathbf{e}^i$ can be defined at each point P_0 of B_0, and covariant and contravariant base vectors $\mathbf{E}_i$, $\mathbf{E}^i$ can be defined at each point P of B.

A stress vector $\mathbf{t}$, measured per unit area of the strained body B, and associated with the element of surface normal to the unit vector $\mathbf{n}$, can be defined at P in the strained body B. It was shown in I that

$$\mathbf{t} = \tau^{ik} n_i \mathbf{E}_k, \tag{2·7}$$

where

$$\mathbf{n} = n_i \mathbf{E}^i,$$

and τ^{ik} are the contravariant components of a symmetrical tensor called the stress tensor.

The stress equations of motion (see I) can be expressed in the form

$$\tau^{ik}\|_i + \rho F^k = \rho f^k, \tag{2·8}$$

where ρ is the density of the strained body B and the double line denotes covariant differentiation with respect to the strained medium B, that is, with respect to θ_i and the metric tensor components G_{ik}, G^{ik}. $\mathbf{F}$ and $\mathbf{f}$ are the body-force and acceleration vectors respectively, and

$$\mathbf{F} = F^k \mathbf{E}_k, \quad \mathbf{f} = f^k \mathbf{E}_k. \tag{2·9}$$

When surface forces are prescribed at a boundary,

$$\mathbf{t} = \mathbf{P}, \tag{2·10}$$

where $\mathbf{P}$ is the surface force measured per unit area of the deformed surface. This may be written in terms of components referred to base vectors $\mathbf{E}_k$ in the form

$$\tau^{ik} n_i = P^k. \tag{2·11}$$

When a strain-energy function W', per unit volume of the unstrained body, exists, it was shown in I that

$$\tau^{ik} = \sqrt{\left(\frac{g}{G}\right)} \frac{\partial W'}{\partial \gamma_{ik}}, \tag{2·12}$$

where

$$G = |G_{ik}|, \quad g = |g_{ik}|.$$

For a material which is isotropic in the undeformed state, the strain-energy function W' is a function of the three strain invariants I_1, I_2 and I_3 which are given by

$$I_1 = g^{rs} G_{rs}, \quad I_2 = g_{rs} G^{rs} I_3, \quad I_3 = \frac{G}{g}. \tag{2·13}$$

It was shown in II that

$$\frac{\partial I_1}{\partial \gamma_{ik}} = 2g^{ik}, \quad \frac{\partial I_2}{\partial \gamma_{ik}} = 2[I_1 g^{ik} - g^{ir} g^{sk} G_{rs}], \quad \frac{\partial G}{\partial \gamma_{ik}} = 2GG^{ik}. \tag{2·14}$$

From (2·13) and (2·14) we see that

$$\frac{\partial I_3}{\partial \gamma_{ik}} = 2I_3 G^{ik}, \tag{2·15}$$

and the substitution of (2·13) and (2·15) in (2·12) leads to the stress-strain relation

$$\tau^{ik} = \Phi g^{ik} + \Psi B^{ik} + pG^{ik}, \tag{2·16}$$

where we have put
$$\Phi = 2I_3^{-\frac{1}{2}}\frac{\partial W'}{\partial I_1}, \quad \Psi = 2I_3^{-\frac{1}{2}}\frac{\partial W'}{\partial I_2},$$
$$B^{ik} = I_1 g^{ik} - g^{ir}g^{sk}G_{rs}, \quad p = 2I_3^{\frac{1}{2}}\frac{\partial W'}{\partial I_3},$$
Φ, Ψ and p being scalar invariant functions of I_1, I_2, I_3. This notation differs slightly from that used in II.

For an incompressible material, the incompressibility condition is $G = g$, or $I_3 = 1$, at all points of the body, and the strain-energy function W' is then a function of I_1 and I_2 only. As shown in II, the stress-strain relation (2·16) is still valid, but in this case we have
$$\Phi = 2\frac{\partial W'}{\partial I_1}, \quad \Psi = 2\frac{\partial W'}{\partial I_2},$$
and p is a scalar invariant function of the co-ordinates θ_i for each value of the time t.

In the latter half of the paper we restrict our discussion to materials which are incompressible and which have the strain-energy function
$$W' = C_1(I_1-3)+C_2(I_2-3) \tag{2·17}$$
used by Mooney (1940) for rubber-like materials, where C_1 and C_2 are constants. For this material, (2·16) gives us the stress-strain relation
$$\tau^{ik} = 2C_1 g^{ik}+2C_2 B^{ik}+pG^{ik}. \tag{2·18}$$
This completes the summary of formulae which will be of use in the present paper.

3. Small twist superposed upon finite extension

We suppose that the unstrained body B_0 is a cylinder of isotropic elastic material of constant cross-section R_0 whose generators are parallel to the x_3-axis, the plane ends of the cylinder being $x_3 = 0$ and $x_3 = l_0$. The cylinder is now given a uniform simple extension λ along the x_3-axis so that the point (x_1, x_2, x_3) moves to the point $(\lambda_1 x_1, \lambda_1 x_2, \lambda x_3)$, where λ_1 is the extension ratio in directions perpendicular to the x_3-axis. We shall take the convected co-ordinates θ_i of a point in the elastic body to be the co-ordinates (x, y, z) of the point after the simple extension has been imposed, referred to Cartesian axes coincident with the x_i-axes.

Then we have
$$x = \lambda_1 x_1, \quad y = \lambda_1 x_2, \quad z = \lambda x_3, \tag{3·1}$$
where we have written (x, y, z) for $(\theta_1, \theta_2, \theta_3)$, and also
$$g_{ik} = \begin{pmatrix} \frac{1}{\lambda_1^2}, & 0, & 0 \\ 0, & \frac{1}{\lambda_1^2}, & 0 \\ 0, & 0, & \frac{1}{\lambda^2} \end{pmatrix}, \quad g^{ik} = \begin{pmatrix} \lambda_1^2, & 0, & 0 \\ 0, & \lambda_1^2, & 0 \\ 0, & 0, & \lambda^2 \end{pmatrix}, \quad g = \frac{1}{\lambda_1^4\lambda^2}. \tag{3·2}$$

Finally, the cylinder is given a small twist ψ, per unit length of the strained body, about the x_3-axis. We assume that the final co-ordinates y_i of a typical point of the strained body are then
$$y_1 = x-\psi yz, \quad y_2 = y+\psi xz, \quad y_3 = z+\psi\phi(x, y), \tag{3·3}$$

where ψ is considered small in the classical sense, $\phi(x,y)$ is the warping function, and the y_i-axes are taken coincident with the x_i-axes. From (3·3) we find that

$$\frac{\partial y_i}{\partial \theta_k} = \begin{pmatrix} 1, & -\psi z, & -\psi y \\ \psi z, & 1, & \psi x \\ \psi\phi_x, & \psi\phi_y, & 1 \end{pmatrix}, \tag{3·4}$$

where suffixes x, y denote partial differentiation with respect to these suffixes.

Neglecting powers of ψ above the first we obtain, from (3·4),

$$G_{ik} = \begin{pmatrix} 1, & 0, & \psi(\phi_x-y) \\ 0, & 1, & \psi(\phi_y+x) \\ \psi(\phi_x-y), & \psi(\phi_y+x), & 1 \end{pmatrix},$$

$$G^{ik} = \begin{pmatrix} 1, & 0, & -\psi(\phi_x-y) \\ 0, & 1, & -\psi(\phi_y+x) \\ -\psi(\phi_x-y), & -\psi(\phi_y+x), & 1 \end{pmatrix}, \quad G = 1. \tag{3·5}$$

The strain invariants (2·13) are found to be

$$I_1 = 2\lambda_1^2+\lambda^2, \quad I_2 = 2\lambda_1^2\lambda^2+\lambda_1^4, \quad I_3 = \lambda_1^4\lambda^2, \tag{3·6}$$

and since the invariants are constants, the functions Φ, Ψ, p are also constants. The tensor components B^{ik} are given by

$$B^{ik} = \begin{pmatrix} \lambda_1^2(\lambda_1^2+\lambda^2), & 0, & -\psi\lambda_1^2\lambda^2(\phi_x-y) \\ 0, & \lambda_1^2(\lambda_1^2+\lambda^2), & -\psi\lambda_1^2\lambda^2(\phi_y+x) \\ -\psi\lambda_1^2\lambda^2(\phi_x-y), & -\psi\lambda_1^2\lambda^2(\phi_y+x), & 2\lambda_1^2\lambda^2 \end{pmatrix}. \tag{3·7}$$

The substitution of (3·2), (3·5) and (3·7) in the stress-strain relation (2·16) leads to the following values of the components of the stress tensor:

$$\left.\begin{aligned} \tau^{11} &= \tau^{22} = \Phi\lambda_1^2+\Psi\lambda_1^2(\lambda_1^2+\lambda^2)+p, \\ \tau^{33} &= \Phi\lambda^2+2\Psi\lambda_1^2\lambda^2+p, \quad \tau^{12} = 0, \\ \tau^{13} &= -\psi(\phi_x-y)\,\{\Psi\lambda_1^2\lambda^2+p\}, \\ \tau^{23} &= -\psi(\phi_y+x)\,\{\Psi\lambda_1^2\lambda^2+p\}. \end{aligned}\right\} \tag{3·8}$$

The components τ^{11} and τ^{22} will be zero if

$$\Phi\lambda_1^2+\Psi\lambda_1^2(\lambda_1^2+\lambda^2)+p = 0, \tag{3·9}$$

and, when the material is compressible, this equation serves to determine λ_1 in terms of λ if the form of the strain-energy function is known. When the material is incompressible, we have $\lambda_1 = 1/\surd\lambda$ since $G = g$ in this case, and equation (3·9) gives the value of the pressure function p.

Using equation (3·9), the non-zero components of the stress tensor can be written

$$\left.\begin{aligned} \tau^{33} &= \{\Phi+\Psi\lambda_1^2\}\,(\lambda^2-\lambda_1^2) = H \quad \text{(say)}, \\ \tau^{13} &= \psi(\phi_x-y)\,\{\Phi+\Psi\lambda_1^2\}\,\lambda_1^2, \\ \tau^{23} &= \psi(\phi_y+x)\,\{\Phi+\Psi\lambda_1^2\}\,\lambda_1^2. \end{aligned}\right\} \tag{3·10}$$

Since we have
$$\left\{\begin{matrix} & i & \\ i & & k\end{matrix}\right\} = \frac{1}{\surd G}\frac{\partial \surd G}{\partial \theta_k} = 0 \quad (k = 1, 2, 3)$$
in this case, where $\left\{\begin{matrix} & i & \\ r & & s\end{matrix}\right\}$ are the Christoffel symbols of the second kind for the co-ordinate system θ_i in the strained body, the equations of equilibrium with no body forces, obtained from (2·8), are
$$\tau^{ik}{}_{,i} + \left\{\begin{matrix} & k & \\ i & & r\end{matrix}\right\}\tau^{ir} = 0,$$
where the comma denotes partial differentiation with respect to θ_i. The Christoffel symbols are of the order of ψ, so that we need only know the symbols $\left\{\begin{matrix} & k & \\ 3 & & 3\end{matrix}\right\}$ to obtain the equations of equilibrium. These symbols are found to be zero so that the first two equations are automatically satisfied, while the third equation gives
$$\phi_{xx} + \phi_{yy} = 0. \tag{3·11}$$

We now consider the boundary conditions. We suppose that the curved surface of the cylinder in the strained state is the surface
$$F(x, y) = 0, \tag{3·12}$$
where the surface
$$F(\lambda_1 x_1, \lambda_1 x_2) = 0 \tag{3·13}$$
was the curved surface of the cylinder in the unstrained state. Equation (3·12) is to be interpreted as the parametric equation of the surface. The co-ordinates y_i of a point on the strained surface are given in terms of the parameters x, y by means of the equations (3·3). The unit normal **n** to the strained surface has covariant (or contravariant) components referred to the y_i-axes which are proportional to $\partial F/\partial y_i$, and a simple tensor transformation shows that the covariant components of **n** referred to the base vectors $\mathbf{E}^i$ defined in § 2 are proportional to
$$\frac{\partial F}{\partial y_i}\frac{\partial y_i}{\partial \theta_k} = \frac{\partial F}{\partial \theta_k},$$
that is,
$$n_1 : n_2 : n_3 = F_x : F_y : 0. \tag{3·14}$$

The first two boundary conditions on the curved surface, which we suppose to be free from applied traction, are automatically satisfied, while the third condition is
$$(\phi_x - y)\, F_x + (\phi_y + x)\, F_y = 0 \quad \text{on} \quad F(x, y) = 0. \tag{3·15}$$

If the harmonic function $w(x_1, x_2)$ is the classical torsion function for the cylinder, then $w(x_1, x_2)$ is such that
$$\left(\frac{\partial w}{\partial x_1} - x_2\right)\frac{\partial F}{\partial x_1} + \left(\frac{\partial w}{\partial x_2} + x_1\right)\frac{\partial F}{\partial x_2} = 0 \quad \text{on} \quad F(\lambda_1 x_1, \lambda_1 x_2) = 0.$$
This condition may be written
$$(\lambda_1 w_x - y/\lambda_1)\, F_x + (\lambda_1 w_y + x/\lambda_1)\, F_y = 0 \quad \text{on} \quad F(x, y) = 0, \tag{3·16}$$
and, by comparison of (3·15) and (3·16), we see that
$$\phi(x, y) = \lambda_1^2 w(x_1, x_2) = \lambda_1^2 w(x/\lambda_1, y/\lambda_1). \tag{3·17}$$

The end $x_3 = l_0$ of the unstrained cylinder becomes the surface $z = l = \lambda l_0$ in the strained state. The unit normal to this surface is $\mathbf{E}^3$, and therefore the tensor components of traction (2·11) on this surface, referred to base vectors $\mathbf{E}_k$, are τ^{3k}. If q^k are the components of surface traction referred to the y_i-axes, then we have

$$q^k = \left\{\tau^{3i}\frac{\partial y_k}{\partial \theta_i}\right\}_{z=l} = (\tau^{13}-\psi y\tau^{33},\ \tau^{23}+\psi x\tau^{33},\ \tau^{33}). \tag{3·18}$$

The element of area of the surface $z = l$ is $dS = \sqrt{(GG^{33})}\,d\theta^1\,d\theta^2 = dx\,dy$, and if Y^k are the components of the resultant force over the end of the cylinder, referred to the y_i-axes, then we find that

$$Y^1 = \iint_R q^1\,dS = -\psi HA_x, \quad Y^2 = \psi HA_y, \tag{3·19}$$

and

$$Y^3 = H\iint_R dx\,dy = HA = \{\Phi+\Psi\lambda_1^2\}(\lambda^2-\lambda_1^2)\,\lambda_1^2 A_0, \tag{3·20}$$

where

$$A_x = \iint_R y\,dx\,dy, \quad A_y = \iint_R x\,dx\,dy,$$

and where A and A_0 are the areas of the cross-sections of the strained and unstrained cylinders respectively, and R is the domain enclosed by the curve $F(x, y) = 0$ in the xy-plane. The transverse forces Y^1, Y^2 are zero if the axis of torsion coincides with the line of centroids of the cross-sections.

The moments m^1, m^2 of the total traction on $z = l$ about axes through the point $y_i = (0, 0, l)$ and parallel to the y_1-axis and the y_2-axis respectively, are given by

$$\begin{aligned} m^1 &= \iint_R \{y_2 q^3-(y_3-l)\,q^2\}_{z=l}\,dS = \iint_R \{y_2 q^3\}_{z=l}\,dx\,dy \\ &= HA_x+\psi lHA_y, \end{aligned} \tag{3·21}$$

$$\begin{aligned} m^2 &= \iint_R \{(y_3-l)\,q^1-y_1 q^3\}_{z=l}\,dS = -\iint_R \{y_1 q^3\}_{z=l}\,dx\,dy \\ &= -HA_y+\psi lHA_x. \end{aligned} \tag{3·22}$$

Remembering that the cross-section $z = l$ has been turned through an angle ψl during the deformation, we see that m^1, m^2 are the moments of the force H extending the cylinder taken about the y_1-, y_2-axes. These moments are zero if the axis of torsion coincides with the line of centroids of the cross-sections.

The moment m^3 of the traction on $z = l$ about the y_3-axis is given by

$$\begin{aligned} m^3 &= \iint_R \{y_1 q^2-y_2 q^1\}_{z=l}\,dS \\ &= \psi\{\Phi+\Psi\lambda_1^2\}\lambda_1^2 S+\psi HI, \end{aligned} \tag{3·23}$$

where

$$\left.\begin{aligned} S &= \iint_R \{x^2+y^2+x\phi_y-y\phi_x\}\,dx\,dy, \\ I &= \iint_R (x^2+y^2)\,dx\,dy. \end{aligned}\right\} \tag{3·24}$$

and

Owing to the term ψHI in the expression for m^3, the magnitude of the twisting couple m^3 depends upon the position of the axis of torsion, which is here taken to be the x_3-axis, and when H is positive the couple is least when the axis passes through the centroids of the cross-sections; that is, when the transverse forces Y^1, Y^2 and the moments m^1, m^2 are zero.

Using (3·1) and (3·17), it can be shown that

$$\left.\begin{aligned} S &= \lambda_1^4 \iint_{R_0} \left\{x_1^2 + x_2^2 + x_1 \frac{\partial w}{\partial x_2} - x_2 \frac{\partial w}{\partial x_1}\right\} dx_1\, dx_2 = \lambda_1^4 S_0, \\ I &= \lambda_1^4 \iint_{R_0} \{x_1^2 + x_2^2\}\, dx_1\, dx_2 = \lambda_1^4 I_0, \end{aligned}\right\} \tag{3·25}$$

where S_0 is the torsional rigidity of the unextended cylinder, and I_0 is the moment of inertia of the unstrained cross-section R_0 about the x_3-axis. The torsional rigidity S_0 differs by a constant factor from the usual definition of the classical torsional rigidity.

By substituting the values of S, I and H given by (3·25) and (3·10) into the expression (3·23) for m^3, we obtain

$$m^3 = \psi\lambda_1^4\{\Phi + \Psi\lambda_1^2\}\{\lambda^2 I_0 - \lambda_1^2(I_0 - S_0)\}. \tag{3·26}$$

It has been shown by Diaz & Weinstein (1948) that $S_0 \leqslant I_0$, and the equality sign holds only when the cross-section is a circular region or a circular ring bounded by two concentric circles. Hence we see from (3·26) that when the cylinder is not a circular cylinder or a circular cylindrical tube, the twisting couple m^3 will be zero, to the first order in ψ at least, when λ is such that

$$\frac{\lambda^2}{\lambda_1^2} = \frac{I_0 - S_0}{I_0} < 1, \tag{3·27}$$

provided that $\lambda_1^4\{\Phi + \Psi\lambda_1^2\}$ is finite for this value of λ.

Although the value of λ_1 is given in terms of λ by (3·9) and cannot be determined explicitly unless the particular form of W' is known, it is to be expected from practical considerations that λ_1 will often be greater than unity when λ is less than unity and the cylinder is compressed in the direction of its length. It is probable, therefore, that a value of λ exists for which (3·27) is true.

When the material is incompressible, $\lambda_1 = 1/\sqrt{\lambda}$ and equation (3·27) becomes

$$\lambda^3 = \frac{I_0 - S_0}{I_0}, \tag{3·28}$$

and the value of λ determined by this equation is independent of the particular form of the strain-energy function which applies to the material. Assuming that the cylinder is twisted about the line joining the centroids of the cross-sections, the value of λ satisfying equation (3·28) for a cross-section which is an ellipse, the major axis of which is twice the minor axis, is 0·71. When the cross-section is an ellipse of which the major axis is four times the minor axis, the corresponding value of λ is 0·92.

From (3·20) and (3·26) we obtain

$$\frac{Y^3}{m^3/\psi} = \frac{(1/\lambda_1^2 - 1/\lambda^2)\, A_0}{\{I_0 - (I_0 - S_0)\, \lambda_1^2/\lambda^2\}}, \tag{3·29}$$

and this relates the force necessary to produce a large simple extension with the torsional modulus for a small twist superposed upon that simple extension. When the material is incompressible, (3·29) becomes

$$\frac{Y^3}{m^3/\psi} = \frac{(\lambda - 1/\lambda^2)\, A_0}{\{I_0 - (I_0 - S_0)/\lambda^3\}}, \tag{3·30}$$

and, in this case, the relation is independent of the particular form of the strain-energy function which applies to the material. The law (3·30) has been obtained previously by Rivlin (1949*c*) for the special case of a circular cylinder, and in this case it has been verified experimentally by Rivlin & Saunders (1951).

Equation (3·29) is expressed in terms of the extension ratio and the dimensions of the unstrained (and unstressed) cylinder, and is analogous to a formula obtained by Biot (1939*b*) and Goodier (1950), but their result is not expressed in terms of a cylinder which is initially unstrained and unstressed.

4. Small twist in presence of hydrostatic pressure

In this section we discuss the effect of a hydrostatic pressure upon the small torsion of the cylinder defined in the previous section. We assume that the uniform pressure ϖ causes a uniform compression λ of the cylinder, and we take our convected co-ordinates θ_i to be given by

$$(\theta_1, \theta_2, \theta_3) = (x, y, z) = (\lambda x_1, \lambda x_2, \lambda x_3). \tag{4·1}$$

This implies that

$$g_{ik} = \frac{1}{\lambda^2}\delta_{ik}, \quad g^{ik} = \lambda^2\delta^{ik}, \quad g = \frac{1}{\lambda^6}, \tag{4·2}$$

where δ_{ik}, δ^{ik} are Kronecker deltas.

As in the previous section, the final co-ordinates y_i of a typical point of the strained body are taken to be

$$y_1 = x - \psi yz, \quad y_2 = y + \psi xz, \quad y_3 = z + \psi\phi(x, y), \tag{4·3}$$

and the metric tensor components G_{ik}, G^{ik} are given by (3·5).

The strain invariants (2·13) are found to be

$$I_1 = 3\lambda^2, \quad I_2 = 3\lambda^4, \quad I_3 = \lambda^6,$$

showing that the functions Φ, Ψ and p are constants.

It is a simple matter to calculate the components of the stress tensor from the stress-strain relation (2·16), and, omitting the details of the calculation, we find that

$$\left.\begin{aligned} &\tau^{11} = \tau^{22} = \tau^{33} = \Phi\lambda^2 + 2\Psi\lambda^4 + p, \quad \tau^{12} = 0, \\ &\tau^{13} = -\psi(\phi_x - y)\{\Psi\lambda^4 + p\}, \quad \tau^{23} = -\psi(\phi_y + x)\{\Psi\lambda^4 + p\}. \end{aligned}\right\} \tag{4·4}$$

Since we suppose that there is a uniform hydrostatic pressure ϖ we require

$$\Phi\lambda^2 + 2\Psi\lambda^4 + p = -\varpi, \tag{4·5}$$

and the stress components (4·4) can then be written in the form

$$\left.\begin{aligned} &\tau^{11} = \tau^{22} = \tau^{33} = -\varpi, \quad \tau^{12} = 0, \\ &\tau^{13} = \psi(\phi_x - y)\{\Phi\lambda^2 + \Psi\lambda^4 + \varpi\}, \\ &\tau^{23} = \psi(\phi_y + x)\{\Phi\lambda^2 + \Psi\lambda^4 + \varpi\}. \end{aligned}\right\} \tag{4·6}$$

Equation (4·5) serves to determine λ for a given pressure ϖ when the form of the strain-energy function is known; alternatively, if the material is incompressible and therefore $\lambda = 1$, the equation gives the value of the pressure function p.

The equations of equilibrium, with no body forces, can be obtained in a similar manner to that of the previous section, and it is found that they will be satisfied, to the first order in ψ, if we have

$$\phi_{xx}+\phi_{yy}=0. \tag{4·7}$$

As in the preceding section, we take the curved surface of the cylinder in the strained state to be the surface (3·12), where

$$F(\lambda x_1, \lambda x_2)=0$$

was the curved surface in the unstrained state. The surface force per unit area of this surface due to the applied hydrostatic pressure is the vector $-\varpi\mathbf{n}$, where $\mathbf{n}$ is the unit normal to the strained surface. Employing (3·14), it is found that the first two boundary conditions, given by (2·11), are automatically satisfied, while the third condition is

$$(\phi_x-y)\,F_x+(\phi_y+x)\,F_y=0 \quad \text{on} \quad F(x,y)=0. \tag{4·8}$$

It can be shown in a similar manner to that of the previous section that we have

$$\phi(x,y)=\lambda^2 w(x_1,x_2)=\lambda^2 w(x/\lambda, y/\lambda), \tag{4·9}$$

where $w(x_1,x_2)$ is the classical torsion function for the cylinder.

On the end $z=l=\lambda l_0$ of the deformed cylinder, the unit normal to which is $\mathbf{E}^3$, the components of surface traction (2·11) are τ^{3k}, and if q^k are the components of surface traction referred to the y_i-axes, then we have, as in § 3,

$$q^k=\left\{\tau^{3i}\frac{\partial y_k}{\partial\theta_i}\right\}_{z=l}=(\tau^{13}-\psi y\tau^{33},\, \tau^{23}+\psi x\tau^{33},\, \tau^{33}). \tag{4·10}$$

The components Y^k of the resultant force over the ends of the cylinder are, in the notation of § 3,

$$Y^1=\psi\varpi A_x, \quad Y^2=-\psi\varpi A_y, \quad Y^3=-\varpi A=-\varpi\lambda^2 A_0, \tag{4·11}$$

while the moment m^3 of the traction on $z=l$ about the y_3-axis is given by

$$\begin{aligned} m^3 &= \iint_R \{y_1 q^2-y_2 q^1\}_{z=l}\,dS \\ &= \psi\{\Phi\lambda^2+\Psi\lambda^4+\varpi\}\,S-\psi\varpi I. \end{aligned} \tag{4·12}$$

The results (3·25), in which λ_1 is replaced by λ, can be used to write (4·12) in the form

$$m^3=\psi\{\Phi+\Psi\lambda^2\}\,\lambda^6 S_0-\psi\varpi\lambda^4(I_0-S_0). \tag{4·13}$$

The couple m^3 will be zero, for cross-sections which are not bounded by concentric circles, when the pressure ϖ is such that the value of λ, determined by equation (4·5), gives

$$\varpi=\frac{\lambda^2\{\Phi+\Psi\lambda^2\}\,S_0}{(I_0-S_0)}. \tag{4·14}$$

For an incompressible material, $\lambda=1$, and the applied couple (4·13) is then

$$\begin{aligned} m^3 &= \psi\{\Phi+\Psi\}_{\lambda=1}\,S_0-\psi\varpi(I_0-S_0) \\ &= \tfrac{1}{3}\psi E S_0-\psi\varpi(I_0-S_0), \end{aligned}$$

where E is Young's modulus of the material for small strains. The twisting couple will be zero if

$$\varpi=\tfrac{1}{3}E\frac{S_0}{I_0-S_0}, \tag{4·15}$$

provided that the cylinder is not a circular cylinder or a circular cylindrical tube, when the torsion couple is unaltered by the presence of the hydrostatic pressure.

The effect of a finite simple extension along its length together with a uniform hydrostatic pressure upon the small torsion of the cylinder can also be determined. It is found that the twisting couple m^3 is given by

$$m^3 = \psi\lambda_1^4\{(\Phi+\Psi\lambda_1^2)\,[\lambda^2 I_0-\lambda_1^2(I_0-S_0)]-\varpi(I_0-S_0)\}, \tag{4·16}$$

where ϖ is the hydrostatic pressure, λ is the extension ratio in the direction of the length of the cylinder and λ_1 is the extension ratio in directions perpendicular to this direction. The extension ratio λ_1 is given, in terms of λ and ϖ, by the equation

$$\Phi\lambda_1^2+\Psi\lambda_1^2(\lambda_1^2+\lambda^2)+p=-\varpi. \tag{4·17}$$

For an incompressible material, $\lambda_1 = 1/\surd\lambda$, and the couple is then

$$m^3 = \psi\{(\Phi+\Psi/\lambda)\,[\lambda^2 I_0-(I_0-S_0)/\lambda]-\varpi(I_0-S_0)\}/\lambda^2. \tag{4·18}$$

PURE TORSION OF INCOMPRESSIBLE CYLINDERS: SECOND-ORDER EFFECTS

5. Statement of the problem

In this part of the paper we shall discuss the secondary effects accompanying the pure torsion of an incompressible cylinder of constant cross-section. The most convenient formulation and solution of the problem is in terms of complex variables, but, for clarity, equations are first derived in terms of real co-ordinates and complex variables are introduced later. The experience gained in considering this problem enables us to use complex variable techniques throughout the formulation and solution of a more general problem. This is considered in the last part of the paper and is the problem of the second-order effects arising from the torsion of an incompressible cylinder which has previously been subjected to a finite extension.

The unstrained body is taken to be a cylinder of constant cross-section R_0 whose generators are parallel to the x_3-axis and the plane ends of which are $x_3 = 0$ and $x_3 = l_0$. The material of which the cylinder is composed is assumed to be incompressible and isotropic in the unstrained state and to have the strain-energy function W' given by (2·17). It has been pointed out by Rivlin & Saunders (1951) that this is the most general form of the strain-energy function W' if, when W' is assumed to be expanded as a double power series in I_1-3 and I_2-3, terms of higher order than the third order of smallness are neglected in the expression for W'; that is, if terms of higher order than the second are neglected in the expressions for the stress components.

We shall take the convected co-ordinates θ_i of a point in the elastic body to be the Cartesian co-ordinates x_i of the unstrained state and we put $\theta_i = x_i = (x, y, z)$. It follows that

$$g^{ik} = \delta^{ik}, \quad g = 1. \tag{5·1}$$

We wish to find a set of displacements whose components u_i along the x_i-axes are such that the cylinder is in a state of torsion about the z-axis (or x_3-axis), and we aim to choose our displacements so that the equilibrium of the cylinder is maintained only by forces at the ends $z = 0$, $z = l_0$, the cylindrical surface of the cylinder being free from applied stress.

If the cross-section is circular, and if, in pure torsion, each section of the cylinder which is normal to the z-axis is rotated through an angle ψz, the displacement components may be written

$$u_1 = (x\cos\psi z - y\sin\psi z) - x, \quad u_2 = (x\sin\psi z + y\cos\psi z) - y, \quad u_3 = 0.$$

These displacements automatically satisfy the incompressibility condition. If powers of ψ above the second are neglected then the above displacements become approximately

$$u_1 = -\psi yz - \tfrac{1}{2}\psi^2 xz^2, \quad u_2 = \psi xz - \tfrac{1}{2}\psi^2 yz^2, \quad u_3 = 0.$$

These forms suggest that for an arbitrary cross-section we should assume that the displacements u_i are

$$\left.\begin{aligned} u_1 &= -\psi yz - \tfrac{1}{2}\psi^2 xz^2 - \tfrac{1}{2}\psi^2 hx + \psi^2 U, \\ u_2 &= \psi xz - \tfrac{1}{2}\psi^2 yz^2 - \tfrac{1}{2}\psi^2 hy + \psi^2 V, \\ u_3 &= \psi\phi(x,y) + \psi^2 hz + \psi^2 W, \end{aligned}\right\} \tag{5·2}$$

where, here and subsequently, all quantities which contain the third and higher powers of ψ are neglected. The functions U, V, W, ϕ are functions of x and y only and ϕ is the classical torsion function. The terms containing the constant h represent an extension of the cylinder of the second order in ψ. Taking the y_i-axes to coincide with the x_i-axes we obtain, since $y_i = x_i + u_i$,

$$\frac{\partial y_i}{\partial \theta_k} = \begin{pmatrix} 1-\frac{1}{2}\psi^2 z^2 - \frac{1}{2}\psi^2 h + \psi^2 U_x, & -\psi z + \psi^2 U_y, & -\psi y - \psi^2 xz \\ \psi z + \psi^2 V_x, & 1 - \frac{1}{2}\psi^2 z^2 - \frac{1}{2}\psi^2 h + \psi^2 V_y, & \psi x - \psi^2 yz \\ \psi\phi_x + \psi^2 W_x, & \psi\phi_y + \psi^2 W_y, & 1 + \psi^2 h \end{pmatrix}, \tag{5·3}$$

and
$$\surd G = 1 + \psi^2(y\phi_x - x\phi_y + U_x + V_y),$$

where, as before, suffixes x, y denote partial differentiation with respect to these suffixes.

The incompressibility condition $G = g$ is therefore

$$U_x + V_y = x\phi_y - y\phi_x. \tag{5·4}$$

The covariant components G_{ik} of the metric tensor are found from (5·3) to be

$$\left.\begin{aligned} G_{11} &= 1 + \psi^2(2U_x - h + \phi_x^2), \\ G_{22} &= 1 + \psi^2(2V_y - h + \phi_y^2), \\ G_{33} &= 1 + \psi^2(2h + x^2 + y^2), \\ G_{12} &= \psi^2(U_y + V_x + \phi_x\phi_y), \\ G_{13} &= \psi(\phi_x - y) + \psi^2 W_x, \\ G_{23} &= \psi(\phi_y + x) + \psi^2 W_y, \end{aligned}\right\} \tag{5·5}$$

and, using the incompressibility condition (5·4), the contravariant components G^{ik} are found to be

$$\left.\begin{aligned} G^{11} &= 1 - \psi^2(2U_x - h + 2y\phi_x - y^2), \\ G^{22} &= 1 - \psi^2(2V_y - h - 2x\phi_y - x^2), \\ G^{33} &= 1 - \psi^2(2h - 2x\phi_y + 2y\phi_x - \phi_x^2 - \phi_y^2), \\ G^{12} &= -\psi^2(U_y + V_x - x\phi_x + y\phi_y + xy), \\ G^{13} &= -\psi(\phi_x - y) - \psi^2 W_x, \\ G^{23} &= -\psi(\phi_y + x) - \psi^2 W_y. \end{aligned}\right\} \tag{5·6}$$

The first strain invariant is

$$I_1 = g^{rs}G_{rs} = 3+\psi^2\{(\phi_x-y)^2+(\phi_y+x)^2\},$$

and the tensor components B^{ik} defined in § 2 are given by

$$\left.\begin{aligned} B^{11} &= 2-\psi^2\{2U_x-h+2y\phi_x-y^2-(\phi_y+x)^2\}, \\ B^{22} &= 2-\psi^2\{2V_y-h-2x\phi_y-x^2-(\phi_x-y)^2\}, \\ B^{33} &= 2-\psi^2\{2h-2x\phi_y+2y\phi_x-\phi_x^2-\phi_y^2\}, \\ B^{12} &= -\psi^2\{U_y+V_x+\phi_x\phi_y\}, \\ B^{13} &= -\psi(\phi_x-y)-\psi^2W_x, \\ B^{23} &= -\psi(\phi_y+x)-\psi^2W_y. \end{aligned}\right\} \tag{5·7}$$

We know that when quantities containing the second and higher powers of ψ are neglected, the problem reduces to the classical torsion problem. The stresses τ^{11}, τ^{22}, τ^{33}, τ^{12} are therefore at least of the order ψ^2 and, remembering the stress-strain relation (2·18), we assume that the pressure function p has the form

$$p = -2(C_1+2C_2)+2\psi^2\{h(C_1+C_2)+\chi(x,y)\}, \tag{5·8}$$

where χ is a function of x and y only and is independent of the constant h. The substitution of (5·1), (5·6), (5·7) and (5·8) in (2·18) leads to the following values of the stress components, correct to the order ψ^2:

$$\left.\begin{aligned} \tau^{11} &= 2\psi^2\{\chi+(C_1+C_2)\,[2U_x+2y\phi_x-y^2]+C_2(\phi_y+x)^2\}, \\ \tau^{22} &= 2\psi^2\{\chi+(C_1+C_2)\,[2V_y-2x\phi_y-x^2]+C_2(\phi_x-y)^2\}, \\ \tau^{33} &= 2\psi^2\{\chi+(C_1+C_2)\,(3h-2x\phi_y+2y\phi_x-\phi_x^2-\phi_y^2)\}, \\ \tau^{12} &= 2\psi^2\{(C_1+C_2)\,[U_y+V_x-x\phi_x+y\phi_y+xy]-C_2(\phi_x-y)\,(\phi_y+x)\}, \\ \tau^{13} &= 2(C_1+C_2)\,\{\psi(\phi_x-y)+\psi^2W_x\}, \\ \tau^{23} &= 2(C_1+C_2)\,\{\psi(\phi_y+x)+\psi^2W_y\}. \end{aligned}\right\} \tag{5·9}$$

Since
$$\left\{\begin{matrix} & i & \\ i & & k \end{matrix}\right\} = \frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial\theta_k} = 0 \quad (k = 1, 2, 3)$$

in this case, where $\left\{\begin{matrix} & i & \\ j & & k \end{matrix}\right\}$ are the Christoffel symbols for the strained body, the equations of equilibrium with no body forces, obtained from (2·8), are

$$\tau^{ik},_i+\left\{\begin{matrix} & k & \\ i & & r \end{matrix}\right\}\tau^{ir} = 0.$$

The Christoffel symbols are of the order of ψ, so that it is only necessary to calculate the symbols $\left\{\begin{matrix} & k & \\ 1 & & 3 \end{matrix}\right\}$, $\left\{\begin{matrix} & k & \\ 2 & & 3 \end{matrix}\right\}$ to obtain the equations of equilibrium, and it is found that they will be satisfied to our degree of approximation if

$$\left.\begin{aligned} \chi_x+(C_1+C_2)\,\nabla_1^2 U &= C_1x+C_2\{x-(\phi_y+x)\,(3+\phi_{xy})-\phi_{xx}(\phi_x-y)\}, \\ \chi_y+(C_1+C_2)\,\nabla_1^2 V &= C_1y+C_2\{y+(\phi_x-y)\,(3-\phi_{xy})-\phi_{yy}(\phi_y+x)\}, \\ \nabla_1^2 W &= 0, \end{aligned}\right\} \tag{5·10}$$

where
$$\nabla_1^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

The incompressibility condition (5·4) and the harmonic property of the torsion function

$$\nabla_1^2 \phi = 0 \tag{5·11}$$

have been used in formulating equations (5·10). It follows from (5·4) and (5·10) that

$$\nabla_1^2 \chi = 2C_1 - 2C_2\{2+\phi_{xx}^2+\phi_{xy}^2\}. \tag{5·12}$$

We now consider the boundary conditions. If the curved surface of the cylinder in the unstrained state is the surface

$$F(x, y) = 0, \tag{5·13}$$

then the curved surface of the strained cylinder is also given by (5·13), where this equation is interpreted as the parametric equation of the surface. The co-ordinates y_i of the strained surface are given in terms of the parameters x, y by means of the equations $y_i = x_i + u_i$. As in § 3, the covariant components, referred to the θ_i-axes, of the unit normal $\mathbf{n}$ to the strained surface are such that

$$n_1 : n_2 : n_3 = F_x : F_y : 0.$$

The boundary conditions (2·11) on the curved surface, which we suppose to be free from traction, therefore reduce to the three conditions

$$\left.\begin{aligned} &\chi F_x + (C_1+C_2)\{[2U_x+y^2]\,F_x + [U_y+V_x-x\phi_x-y\phi_y-xy]\,F_y\} \\ &\qquad\qquad + C_2(\phi_y+x)\,\{(\phi_y+x)\,F_x-(\phi_x-y)\,F_y\} = 0, \\ &\chi F_y + (C_1+C_2)\{[2V_y+x^2]\,F_y + [U_y+V_x+x\phi_x+y\phi_y-xy]\,F_x\} \\ &\qquad\qquad + C_2(\phi_x-y)\,\{(\phi_x-y)\,F_y-(\phi_y+x)\,F_x\} = 0, \\ &\qquad\qquad W_xF_x+W_yF_y = 0, \end{aligned}\right\} \tag{5·14}$$

on the surface (5·13), where we have used the boundary condition satisfied by the classical torsion function ϕ, i.e.

$$(\phi_x-y)\,F_x+(\phi_y+x)\,F_y = 0 \tag{5·15}$$

on the surface (5·13). The third boundary condition in (5·14) is equivalent to $\partial W/\partial n = 0$ at the boundary, i.e. the outward normal derivative of W is zero, and as W is single-valued, and the third equation in (5·10) shows that W is a plane harmonic function, it must therefore be a constant. This constant may be taken to be zero since it represents a rigid-body displacement.

The components τ^{ik} of the stress tensor referred to the θ_i-axes are given by the formulae (5·9) in which W is put equal to zero. Alternatively, we may refer the stresses to the y_i-axes. Denoting the components of the stress tensor referred to the y_i-axes by t^{rs}, we have

$$t^{rs} = \tau^{ik}\frac{\partial y_r}{\partial \theta_i}\frac{\partial y_s}{\partial \theta_k},$$

and (5·3), (5·9) give

$$\left.\begin{aligned} t^{11} &= 2\psi^2\{\chi+(C_1+C_2)\,[2U_x+y^2]+C_2(\phi_y+x)^2\}, \\ t^{22} &= 2\psi^2\{\chi+(C_1+C_2)\,[2V_y+x^2]+C_2(\phi_x-y)^2\}, \\ t^{33} &= 2\psi^2\{\chi+(C_1+C_2)\,(3h+\phi_x^2+\phi_y^2)\}, \\ t^{12} &= 2\psi^2\{(C_1+C_2)\,[U_y+V_x-xy]-C_2(\phi_x-y)\,(\phi_y+x)\}, \\ t^{13} &= 2(C_1+C_2)\,\{\psi(\phi_x-y)-\psi^2z(\phi_y+x)\}, \\ t^{23} &= 2(C_1+C_2)\,\{\psi(\phi_y+x)+\psi^2z(\phi_x-y)\}. \end{aligned}\right\} \tag{5·16}$$

We shall now consider the traction on the end $z = l_0$ of the cylinder which must be applied to maintain the state of stress represented by (5·9) or (5·16). The unit normal to the surface $z = l_0$ in the strained state has covariant components, referred to the base vectors $\mathbf{E}^i$, which are equal to (0, 0, 1) if ψ^2 and higher powers of ψ are neglected. Hence the components of surface traction on $z = l_0$, referred to the base vectors $\mathbf{E}_i$, are τ^{3i}, and if q^k are the components of surface traction referred to the y_i-axes

$$q^k = \left\{\tau^{3i}\frac{\partial y_k}{\partial \theta_i}\right\}_{z=l_0}.$$

That is, to our order of approximation,

$$q^1 = \{t^{13}\}_{z=l_0}, \quad q^2 = \{t^{23}\}_{z=l_0}, \quad q^3 = 2\psi^2\{\chi + (C_1+C_2)\,(3h - x\phi_y + y\phi_x)\}. \qquad (5·17)$$

The element of area on the surface $z = l_0$ in the strained state is $\surd(GG^{33})\,d\theta^1\,d\theta^2$, and this is equal to $dx\,dy$ if we neglect the second and higher powers of ψ. If Y^k are the components of the resultant force over the end of the cylinder, referred to the y_i-axes, it can be verified that

$$Y^\alpha = \iint_{R_0} q^\alpha\,dx\,dy = 0 \quad (\alpha = 1, 2),$$

where R_0 is the domain bounded by the curve $F(x, y) = 0$ in the xy-plane. The resultant force Y^3 parallel to the y_3-axis is

$$Y^3 = \iint_{R_0} q^3\,dx\,dy = 2\psi^2 \iint_{R_0} \{\chi + (C_1+C_2)\,(3h - x\phi_y + y\phi_x)\}\,dx\,dy,$$

and this can also be written in the form

$$Y^3 = 2\psi^2 \iint_{R_0} \{\chi + (C_1+C_2)\,(3h + \phi_x^2 + \phi_y^2)\}\,dx\,dy, \qquad (5·18)$$

for, if we apply Stokes's theorem and use the boundary condition (5·15) together with (5·11), we find that

$$\iint_{R_0} \left\{\frac{\partial}{\partial x}[\phi(\phi_x - y)] + \frac{\partial}{\partial y}[\phi(\phi_y + x)]\right\}dx\,dy = \iint_{R_0} \{\phi_x^2 + \phi_y^2 + x\phi_y - y\phi_x\}\,dx\,dy = 0. \qquad (5·19)$$

The moments m^i of the total traction on $z = l_0$ about axes parallel to the y_i-axes through the point $y_i = (0, 0, l_0)$ are given by

$$\left.\begin{aligned}
m^1 &= \iint_{R_0} \{y_2 q^3 - (y_3 - l_0)\,q^2\}_{z=l_0}\,dx\,dy \\
&= 2\psi^2 \iint_{R_0} \{y\chi + (C_1+C_2)\,[y(3h - x\phi_y + y\phi_x) - \phi(\phi_y + x)]\}\,dx\,dy, \\
m^2 &= \iint_{R_0} \{(y_3 - l_0)\,q^1 - y_1 q^3\}_{z=l_0}\,dx\,dy \\
&= -2\psi^2 \iint_{R_0} \{x\chi + (C_1+C_2)\,[x(3h - x\phi_y + y\phi_x) - \phi(\phi_x - y)]\}\,dx\,dy, \\
m^3 &= \iint_{R_0} \{y_1 q^2 - y_2 q^1\}_{z=l_0}\,dx\,dy \\
&= 2\psi(C_1+C_2) \iint_{R_0} \{x^2 + y^2 + x\phi_y - y\phi_x\}\,dx\,dy \\
&= 2\psi(C_1+C_2)\,S_0,
\end{aligned}\right\} \qquad (5·20)$$

where S_0 is the geometrical torsional rigidity of the unstrained cross-section of the cylinder. The value of m^3 in (5·20) can also be obtained from the general formula (3·26) by putting $\lambda = \lambda_1 = 1$, $\Phi = 2C_1$, $\Psi = 2C_2$.

6. Effect of change of axis of torsion

In the preceding section, the axis of torsion was assumed to be the x_3-axis. If we had taken the axis of torsion to be the line $x_1 = -a$, $x_2 = -b$, then we would have assumed that the displacements u_i' along the x_i-axes were the expressions

$$\left.\begin{aligned} u_1' &= -\psi(y+b)\,z - \tfrac{1}{2}\psi^2(x+a)\,z^2 - \tfrac{1}{2}\psi^2 hx + \psi^2 U', \\ u_2' &= \psi(x+a)\,z - \tfrac{1}{2}\psi^2(y+b)\,z^2 - \tfrac{1}{2}\psi^2 hy + \psi^2 V', \\ u_3' &= \psi\phi'(x,y) + \psi^2 hz, \end{aligned}\right\} \tag{6·1}$$

where U', V', ϕ' are functions of x and y only. We have omitted the third displacement function W, since we found in § 5 that it could be taken to be zero. The incompressibility condition is now

$$U_x' + V_y' = (x+a)\,\phi_y' - (y+b)\,\phi_x',$$

and, as before, the pressure function p occurring in the stress-strain relation (2·18) is assumed to be

$$p = -2(C_1 + 2C_2) + 2\psi^2\{h(C_1 + C_2) + \chi'(x,y)\}.$$

Using the values (6·1) for the displacements and this value of the pressure function p, the expressions for the stresses τ'^{ik} and the equations satisfied by the functions U', V', ϕ', χ' are given by the corresponding expressions and equations of § 5 provided that we replace x, y, U, V, ϕ, χ by $x+a$, $y+b$, U', V', ϕ', χ' respectively.

Thus the stress components referred to the convected co-ordinate system $\theta_i = (x, y, z)$ are now given by

$$\left.\begin{aligned} \tau'^{11} &= 2\psi^2\{\chi' + (C_1+C_2)\,[2U_x' + 2(y+b)\,\phi_x' - (y+b)^2] + C_2(\phi_y' + x + a)^2\}, \\ \tau'^{22} &= 2\psi^2\{\chi' + (C_1+C_2)\,[2V_y' - 2(x+a)\,\phi_y' - (x+a)^2] + C_2(\phi_x' - y - b)^2\}, \\ \tau'^{33} &= 2\psi^2\{\chi' + (C_1+C_2)\,[3h - 2(x+a)\,\phi_y' + 2(y+b)\,\phi_x' - \phi_x'^2 - \phi_y'^2]\}, \\ \tau'^{12} &= 2\psi^2\{(C_1+C_2)\,[U_y' + V_x' - (x+a)\,\phi_x' + (y+b)\,\phi_y' + (x+a)\,(y+b)] \\ &\qquad - C_2(\phi_x' - y - b)\,(\phi_y' + x + a)\}, \\ \tau'^{13} &= 2\psi(C_1+C_2)\,(\phi_x' - y - b), \\ \tau'^{23} &= 2\psi(C_1+C_2)\,(\phi_y' + x + a). \end{aligned}\right\} \tag{6·2}$$

The torsion function ϕ' of this section is harmonic in the cross-section of the cylinder and satisfies the condition

$$(\phi_x' - y - b)\,F_x + (\phi_y' + x + a)\,F_y = 0 \tag{6·3}$$

on the boundary of the cylinder $F(x,y) = 0$. Comparison of (5·15) and (6·3) shows that we must have

$$\phi'(x,y) = \phi(x,y) + bx - ay, \tag{6·4}$$

apart from a non-essential constant. Using the result (6·4) we find that

$$\left.\begin{aligned} U_x' + V_y' &= (x+a)\,(\phi_y - a) - (y+b)\,(\phi_x + b), \\ \chi_x' + (C_1+C_2)\,\nabla_1^2 U' &= C_1(x+a) + C_2\{(x+a) - (\phi_y + x)\,(3 + \phi_{xy}) - \phi_{xx}(\phi_x - y)\}, \\ \chi_y' + (C_1+C_2)\,\nabla_1^2 V' &= C_1(y+b) + C_2\{(y+b) + (\phi_x - y)\,(3 - \phi_{xy}) - \phi_{yy}(\phi_y + x)\}, \\ \nabla_1^2 \chi' &= 2C_1 - 2C_2\{2 + \phi_{xx}^2 + \phi_{xy}^2\}, \end{aligned}\right\} \tag{6·5}$$

with the conditions

$$\left.\begin{aligned}\chi' F_x+(C_1+C_2)\{[2U'_x+(y+b)^2]\,F_x+[U'_y+V'_x-(x+a)\,(\phi_x+b)\\ -(y+b)\,(\phi_y-a)-(x+a)\,(y+b)]\,F_y\}+C_2(\phi_y+x)\,\{(\phi_y+x)\,F_x-(\phi_x-y)\,F_y\}=0,\\ \chi' F_y+(C_1+C_2)\{[2V'_y+(x+a)^2]\,F_y+[U'_y+V'_x+(x+a)\,(\phi_x+b)\\ +(y+b)\,(\phi_y-a)-(x+a)\,(y+b)]\,F_x\}+C_2(\phi_x-y)\,\{(\phi_x-y)\,F_y-(\phi_y+x)\,F_x\}=0,\end{aligned}\right\} \quad (6{\cdot}6)$$

on the boundary $F(x,y)=0$.

We now put

$$U'=U+u,\quad V'=V+v,\quad \chi'=\chi+\xi, \quad (6{\cdot}7)$$

where U, V, χ are the two displacement functions and the pressure function when the axis of torsion is the x_3-axis, that is, when $a=b=0$. By subtracting the equations (6·5) and the boundary conditions (6·6) from the corresponding equations and boundary conditions satisfied by the functions U, V, χ in § 5, we find that u, v, ξ are such that

$$\left.\begin{aligned}u_x+v_y&=a\phi_y-b\phi_x-ax-by-a^2-b^2,\\ \xi_x+(C_1+C_2)\,\nabla_1^2 u&=(C_1+C_2)\,a,\\ \xi_y+(C_1+C_2)\,\nabla_1^2 v&=(C_1+C_2)\,b,\\ \nabla_1^2\xi&=0,\end{aligned}\right\} \quad (6{\cdot}8)$$

with the conditions

$$\left.\begin{aligned}\xi F_x+(C_1+C_2)\,\{(2u_x+2by+b^2)\,F_x+(u_y+v_x-a\phi_x-b\phi_y-2bx-ab)\,F_y\}=0,\\ \xi F_y+(C_1+C_2)\,\{(2v_y+2ax+a^2)\,F_y+(u_y+v_x+a\phi_x+b\phi_y-2ay-ab)\,F_x\}=0,\end{aligned}\right\} \quad (6{\cdot}9)$$

on $F(x,y)=0$.

It can be shown that the solution of the equations (6·8) subject to the boundary conditions (6·9) is given by

$$\left.\begin{aligned}u&=-b\phi-\tfrac{1}{4}a(x^2-y^2)-\tfrac{1}{2}bxy-\tfrac{1}{4}(a^2+3b^2)\,x+\tfrac{1}{2}aby,\\ v&=a\phi-\tfrac{1}{4}b(y^2-x^2)-\tfrac{1}{2}axy-\tfrac{1}{4}(3a^2+b^2)\,y+\tfrac{1}{2}abx,\\ \xi&=(C_1+C_2)\,\{ax+by+\tfrac{1}{2}(a^2+b^2)\},\end{aligned}\right\} \quad (6{\cdot}10)$$

apart from non-essential constants in the expressions for u and v. The formulae (6·7) and (6·10) together give the values of the functions U', V', χ', and the substitution of these values and the value of ϕ' given by (6·4) into the expressions (6·2) for the stress components gives

$$\left.\begin{aligned}\tau'^{11}&=2\psi^2\{\chi+(C_1+C_2)\,[2U_x+2y\phi_x-y^2]+C_2(\phi_y+x)^2\},\\ \tau'^{22}&=2\psi^2\{\chi+(C_1+C_2)\,[2V_y-2x\phi_y-x^2]+C_2(\phi_x-y)^2\},\\ \tau'^{33}&=2\psi^2\{\chi+(C_1+C_2)\,[3h-2x\phi_y+2y\phi_x-\phi_x^2-\phi_y^2+3ax+3by+\tfrac{3}{2}(a^2+b^2)]\},\\ \tau'^{12}&=2\psi^2\{(C_1+C_2)\,[U_y+V_x-x\phi_x+y\phi_y+xy]-C_2(\phi_x-y)\,(\phi_y+x)\},\\ \tau'^{13}&=2\psi(C_1+C_2)\,(\phi_x-y),\\ \tau'^{23}&=2\psi(C_1+C_2)\,(\phi_y+x).\end{aligned}\right\} \quad (6{\cdot}11)$$

If we compare the expressions (6·11) with the expressions (5·9), in which W is put equal to zero, we see that the only component of the stress tensor (referred to the convected coordinate system $\theta_i=(x,y,z)$) which is altered by changing the position of the axis of torsion is the component τ^{33}.

The components q'^k of surface traction on the end $z = l_0$ of the cylinder, referred to the y_i-axes, can be found as in § 5. Thus

$$q'^1 = q^1, \quad q'^2 = q^2,$$
$$q'^3 = 2\psi^2\{\chi + (C_1+C_2)\,[3h - x\phi_y + y\phi_x + 3ax + 3by + \tfrac{3}{2}(a^2+b^2) + b(\phi_x - y) - a(\phi_y + x)]\},$$

where q^1 and q^2 are given by (5·17). The components of resultant force on $z = l_0$ in the y_1, y_2 directions are zero as before, while the third component Y'^3 in the y_3 direction is

$$Y'^3 = \iint_{R_0} q'^3\,dS$$
$$= 2\psi^2 \iint_{R_0} \{\chi + (C_1+C_2)\,[3h - x\phi_y + y\phi_x + 3ax + 3by + \tfrac{3}{2}(a^2+b^2)]\}\,dx\,dy.$$

We can write this in the form

$$Y'^3 = Y^3 + 3\psi^2(C_1+C_2)\,\{2aA_{0y} + 2bA_{0x} + (a^2+b^2)\,A_0\}, \tag{6·12}$$

where Y^3 is given by (5·18) and where

$$A_0 = \iint_{R_0} dx\,dy, \quad A_{0x} = \iint_{R_0} y\,dx\,dy, \quad A_{0y} = \iint_{R_0} x\,dx\,dy.$$

The expression (6·12) shows that the total normal force over the end of the cylinder is least when the axis of torsion coincides with the line of centroids of the cross-section. For, on differentiating Y'^3 partially with respect to a and with respect to b and then putting the derivatives equal to zero, we obtain

$$a = -\frac{A_{0y}}{A_0} = -\bar{x}_1, \quad b = -\frac{A_{0x}}{A_0} = -\bar{x}_2,$$

where $\bar{x}_1$, $\bar{x}_2$ are the co-ordinates of the centroid of the cross-section, showing that Y'^3 is least when the axis of torsion passes through the centroid of the cross-section.

Of the moments m'^k of the total traction on $z = l_0$ about axes parallel to the y_i-axes through the point $y_i = (0, 0, l_0)$, the moment m'^3 is alone unaltered. We have, in fact,

$$m'^1 = \iint_{R_0} \{y_2 q'^3 - (y_3 - l_0)\,q'^2\}_{z=l_0}\,dS$$
$$= 2\psi^2 \iint_{R_0} \{y\chi + (C_1+C_2)\,[y\{3h - x\phi_y + y\phi_x + 3ax + 3by + \tfrac{3}{2}(a^2+b^2)\}$$
$$- \phi(\phi_y + x) + b\{y(\phi_x - y) - x(\phi_y + x)\}]\}\,dx\,dy.$$

This can be written in the form

$$m'^1 = m^1 + 2\psi^2(C_1+C_2)\,\{3aI_{0xy} + 3bI_{0x} + \tfrac{3}{2}(a^2+b^2)\,A_{0x} - bS_0\}, \tag{6·13}$$

where m^1 and S_0 are given by (5·20) and where

$$I_{0xy} = \iint_{R_0} xy\,dx\,dy, \quad I_{0x} = \iint_{R_0} y^2\,dx\,dy.$$

In the same way we find that

$$m'^2 = m^2 - 2\psi^2(C_1+C_2)\,\{3aI_{0y} + 3bI_{0xy} + \tfrac{3}{2}(a^2+b^2)\,A_{0y} - aS_0\}, \tag{6·14}$$

where
$$I_{0y} = \iint_{R_0} x^2\,dx\,dy,$$

and m^2 is given by (5·20). Values a, b may exist which make the moments m'^1, m'^2 zero, and, if this is possible, we can then choose the axis of torsion so that the total traction on $z = l_0$ is equivalent to a force along the y_3-axis and passing through the origin of co-ordinates, together with a couple whose axis is parallel to the y_3-axis. It will also be possible to make the force along the y_3-axis zero by taking a particular value for the constant h.

7. Solution of problem using complex variable

The equations (5·4), (5·10) and (5·12), subject to the boundary conditions (5·14), determine the functions U, V, χ for a given cross-section, which we shall assume is a simply-connected domain bounded by a single closed curve.* The determination of these functions is greatly simplified by using the complex variables $\zeta = x+iy$, $\bar{\zeta} = x-iy$ instead of the real variables x, y.

We shall write
$$D(\zeta,\bar{\zeta}) = U+iV, \quad \phi(x,y) = \tfrac{1}{2}\{f(\zeta)+\bar{f}(\bar{\zeta})\}, \tag{7·1}$$
where $f(\zeta)$ is the classical complex torsion function, and as usual we shall denote the complex conjugate of a quantity by a bar. The complex torsion function is regular in the cross-section R_0 of the cylinder and satisfies the boundary condition
$$\frac{\partial G}{\partial \zeta}\{\bar{f}'(\bar{\zeta})+i\zeta\}+\frac{\partial G}{\partial \bar{\zeta}}\{f'(\zeta)-i\bar{\zeta}\} = 0 \quad \text{or} \quad f(\zeta)-\bar{f}(\bar{\zeta}) = i\zeta\bar{\zeta}, \tag{7·2}$$
on the boundary of the cross-section,
$$F(x,y) \equiv G(\zeta,\bar{\zeta}) = 0, \quad \frac{\partial G}{\partial \zeta}d\zeta+\frac{\partial G}{\partial \bar{\zeta}}d\bar{\zeta} = 0, \tag{7·3}$$
where G is a real function of ζ and $\bar{\zeta}$.

Equation (5·12) is equivalent to
$$4\frac{\partial^2 \chi}{\partial \zeta \partial \bar{\zeta}} = 2C_1-2C_2\{2+f''(\zeta)\bar{f}''(\bar{\zeta})\},$$
and we shall write the solution in the form
$$2\chi = C_1\zeta\bar{\zeta}-C_2\{2\zeta\bar{\zeta}+f'(\zeta)\bar{f}'(\bar{\zeta})\}+2i(C_1+C_2)\{f(\zeta)-\bar{f}(\bar{\zeta})\}$$
$$-iC_2\{\zeta f'(\zeta)-\bar{\zeta}\bar{f}'(\bar{\zeta})\}-\Omega(\zeta)-\bar{\Omega}(\bar{\zeta}), \tag{7·4}$$
in order to simplify the boundary conditions. The function $\Omega(\zeta)$ is a regular function of ζ in the domain R_0 and is undetermined to the extent of a purely imaginary constant. This constant does not affect the stresses, however, and therefore represents a rigid body motion. The first two equations in (5·10) combine to give
$$4(C_1+C_2)\frac{\partial^2 D}{\partial \zeta \partial \bar{\zeta}} = -2\frac{\partial \chi}{\partial \bar{\zeta}}+C_1\zeta+C_2[\zeta+3i\{\bar{f}'(\bar{\zeta})+i\zeta\}-\bar{f}''(\bar{\zeta})\{f'(\zeta)-i\bar{\zeta}\}]$$
$$= \bar{\Omega}'(\bar{\zeta})+2i(C_1+2C_2)\bar{f}'(\bar{\zeta}),$$
so that
$$4(C_1+C_2)D = 2i(C_1+2C_2)\zeta\bar{f}(\bar{\zeta})+\zeta\bar{\Omega}(\bar{\zeta})+\bar{\omega}(\bar{\zeta})+\kappa(\zeta)+C_2\bar{g}(\bar{\zeta}), \tag{7·5}$$

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* We can show that the solution is unique except for a displacement $U = ky+a$, $V = -kx+b$ which does not affect the stresses.

where $\omega(\zeta)$ and $\kappa(\zeta)$ are regular functions of ζ in the domain R_0. The term $C_2\bar{g}(\bar{\zeta})$, where

$$g(\zeta) = \int^{\zeta} \{f'(\sigma)\}^2 \, d\sigma, \tag{7·6}$$

has been added to simplify the boundary conditions.

The incompressibility condition (5·4) gives

$$\frac{\partial D}{\partial \zeta} + \frac{\partial \bar{D}}{\partial \bar{\zeta}} = \frac{i}{2}\{\zeta f'(\zeta) - \bar{\zeta}\bar{f}'(\bar{\zeta})\},$$

that is, using (7·5),

$$-2i(C_1+2C_2)\{f(\zeta) - \bar{f}(\bar{\zeta})\} + \Omega(\zeta) + \bar{\Omega}(\bar{\zeta}) + \kappa'(\zeta) + \bar{\kappa}'(\bar{\zeta}) = 2i(C_1+C_2)\{\zeta f'(\zeta) - \bar{\zeta}\bar{f}'(\bar{\zeta})\}.$$

This equation implies that

$$\kappa'(\zeta) = -\Omega(\zeta) + 2i(C_1+2C_2) f(\zeta) + 2i(C_1+C_2)\,\zeta f'(\zeta) + i\beta, \tag{7·7}$$

where β is a real constant. The displacements arising from the constant β do not contribute to the stresses, however, and may therefore be omitted. Integration of (7·7) gives

$$\kappa(\zeta) = -\Theta(\zeta) + 2i(C_1+C_2)\,\zeta f(\zeta) + 2iC_2 k(\zeta), \tag{7·8}$$

where

$$\Theta(\zeta) = \int^{\zeta} \Omega(\sigma)\, d\sigma, \quad k(\zeta) = \int^{\zeta} f(\sigma)\, d\sigma. \tag{7·9}$$

We omit the details of the calculation, but it can be shown that the boundary conditions (5·14) can be reduced to the condition

$$\frac{\partial G}{\partial \zeta}\{\zeta\bar{\Omega}'(\bar{\zeta}) + \bar{\omega}'(\bar{\zeta}) + (C_1+2C_2)\,\zeta^2\} - \frac{\partial G}{\partial \bar{\zeta}}\{\Omega(\zeta) + \bar{\Omega}(\bar{\zeta}) + 2(C_1+2C_2)\,\zeta\bar{\zeta}\} = 0 \tag{7·10}$$

on $G(\zeta, \bar{\zeta}) = 0$. To obtain this condition, the boundary conditions (7·2) on the torsion function $f(\zeta)$ and the expressions (7·4), (7·5) for χ and D are used. The elastic moduli C_1, C_2 may be removed from the boundary condition (7·10) by writing

$$\Omega(\zeta) = (C_1+2C_2)\,\Gamma(\zeta), \quad \omega(\zeta) = (C_1+2C_2)\,\gamma(\zeta), \tag{7·11}$$

and we obtain

$$\frac{\partial G}{\partial \zeta}\{\zeta\bar{\Gamma}'(\bar{\zeta}) + \bar{\gamma}'(\bar{\zeta}) + \zeta^2\} - \frac{\partial G}{\partial \bar{\zeta}}\{\Gamma(\zeta) + \bar{\Gamma}(\bar{\zeta}) + 2\zeta\bar{\zeta}\} = 0 \tag{7·12}$$

on $G(\zeta, \bar{\zeta}) = 0$.

With (7·3), the condition (7·12) can be written

$$\{\zeta\bar{\Gamma}'(\bar{\zeta}) + \bar{\gamma}'(\bar{\zeta}) + \zeta^2\}\, d\bar{\zeta} + \{\Gamma(\zeta) + \bar{\Gamma}(\bar{\zeta}) + 2\zeta\bar{\zeta}\}\, d\zeta = 0$$

on $G(\zeta, \bar{\zeta}) = 0$, and this is

$$d\{\zeta\bar{\Gamma}(\bar{\zeta}) + \bar{\gamma}(\bar{\zeta}) + \int^{\zeta} \Gamma(\sigma)\, d\sigma + \zeta^2\bar{\zeta}\} = 0$$

on $G(\zeta, \bar{\zeta}) = 0$. Thus the boundary condition (7·12) is equivalent to the condition

$$\zeta\bar{\Gamma}(\bar{\zeta}) + \bar{\gamma}(\bar{\zeta}) + \int^{\zeta} \Gamma(\sigma)\, d\sigma + \zeta^2\bar{\zeta} = 0 \tag{7·12a}$$

on $G(\zeta, \bar{\zeta}) = 0$, where we have absorbed the constant of integration into the integral on the left-hand side.

For some cross-sections the function $G(\zeta, \bar{\zeta})$ may be such that it is possible to determine the functions $\Gamma(\zeta)$, $\gamma(\zeta)$ from the boundary conditions (7·12) or (7·12*a*) by expanding them in a power series or otherwise, but for other cross-sections a more powerful technique is needed. We suppose that the domain R_0 in the ζ-plane, representing the cross-section of the cylinder, is mapped conformally on the interior of the unit circle in the t-plane by the transformation*

$$\zeta = m(t). \tag{7·13}$$

The boundary C_0 of the domain R_0 is mapped on the unit circle $|t| = 1$, which we shall denote by γ, so that we may put

$$G(\zeta, \bar{\zeta}) = G_0(t, \bar{t}) = t\bar{t} - 1, \tag{7·14}$$

and it follows that

$$\frac{\partial G}{\partial \zeta} = \frac{\bar{t}}{m'(t)}, \quad \frac{\partial G}{\partial \bar{\zeta}} = \frac{t}{\bar{m}'(\bar{t})}.$$

Functions of ζ which are regular in the domain R_0 become functions of the variable t regular in the unit circle, and we shall use the notation

$$H(\zeta) = H\{m(t)\} = H_0(t) \tag{7·15}$$

in the subsequent work, where $H(\zeta)$ denotes any function of ζ.

The boundary condition (7·12), after multiplication by $m'(t)\,\bar{m}'(\bar{t})$, is

$$\bar{t}[m(t)\,\bar{\Gamma}_0'(\bar{t}) + \bar{\gamma}_0'(\bar{t}) + \bar{m}'(\bar{t})\,\{m(t)\}^2] - m'(t)\,t[\Gamma_0(t) + \bar{\Gamma}_0(\bar{t}) + 2m(t)\,\bar{m}(\bar{t})] = 0$$

on $t\bar{t} = 1$. This can be written

$$m'(t)\,\Gamma_0(t) - \bar{\gamma}_0'(1/t)/t^2 + \frac{d}{dt}[m(t)\,\bar{\Gamma}_0(1/t) + \bar{m}(1/t)\,\{m(t)\}^2] = 0 \tag{7·16}$$

on $t\bar{t} = 1$, and (7·16) implies that we must also have

$$\gamma_0'(t) - \bar{m}'(1/t)\,\bar{\Gamma}_0(1/t)/t^2 + \frac{d}{dt}[\bar{m}(1/t)\,\Gamma_0(t) + m(t)\,\{\bar{m}(1/t)\}^2] = 0 \tag{7·17}$$

on $t\bar{t} = 1$, since (7·17) is the complex conjugate of (7·16).

We can now reduce the determination of the functions $\Gamma_0(t)$, $\gamma_0(t)$ to the solving of a certain integral equation by a method used by Muschelisvili (1932, 1933) to solve elastic problems in two dimensions.

Certain conditions must be satisfied in order that the operations which are used shall be valid, and we shall suppose, without further repetition, that the various functions which are involved satisfy sufficient conditions for our purpose. For example, we shall use Cauchy's integral formula

$$\frac{1}{2\pi i}\int_{C_0} \frac{\alpha(\zeta)}{\zeta - \sigma}\,d\zeta = \alpha(\sigma),$$

which is valid if $\zeta = \sigma$ is an interior point of the region R_0, whenever $\alpha(\zeta)$ is continuous in the closed region R_0 bounded by a contour C_0 and regular at every interior point of R_0. We shall also use Harnack's theorem which states that if $f_1(\theta)$, $f_2(\theta)$, $\phi_1(\theta)$, $\phi_2(\theta)$ are four real continuous functions of the argument θ (defined on the unit circle γ), and if

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f_1 \pm i f_2}{\zeta - \sigma}\,d\zeta = \frac{1}{2\pi i}\int_{\gamma} \frac{\phi_1 \pm i\phi_2}{\zeta - \sigma}\,d\zeta$$

for all values of σ interior to γ, then

$$\phi_1(\theta) \equiv f_1(\theta), \quad \phi_2(\theta) \equiv f_2(\theta).$$

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* The complex variable t is not to be confused with the time t.

If we multiply (7·16) and (7·17) by $dt/2\pi i(t-\sigma)$, where σ lies inside the unit circle, and integrate around the unit circle γ, we obtain

$$m'(\sigma)\,\Gamma_0(\sigma)+\frac{1}{2\pi i}\int_\gamma m(t)\,\{\overline{\Gamma}_0(1/t)+m(t)\,\overline{m}(1/t)\}\,\frac{dt}{(t-\sigma)^2}=0, \tag{7·18}$$

$$\gamma_0'(\sigma)=-\frac{1}{2\pi i}\int_\gamma \overline{m}(1/t)\,\{\Gamma_0(t)+m(t)\,\overline{m}(1/t)\}\,\frac{dt}{(t-\sigma)^2}, \tag{7·19}$$

where we have used Cauchy's theorem, the rule of integration by parts and the results

$$\frac{1}{2\pi i}\int_\gamma \overline{\gamma}_0'(1/t)\,\frac{dt}{t^2(t-\sigma)}=\frac{1}{2\pi i}\int_\gamma \overline{m}'(1/t)\,\overline{\Gamma}_0(1/t)\,\frac{dt}{t^2(t-\sigma)}=0.$$

By Harnack's theorem, under suitable conditions, (7·18) and (7·19) are completely equivalent to (7·16) and (7·17).

The integral equations (7·18) and (7·20) can also be obtained from the alternative form of the boundary condition (7·12*a*).

Integration of (7·19) with respect to σ gives

$$\gamma_0(\sigma)=-\frac{1}{2\pi i}\int_\gamma \overline{m}(1/t)\,\{\Gamma_0(t)+m(t)\,\overline{m}(1/t)\}\,\frac{dt}{(t-\sigma)}, \tag{7·20}$$

apart from a non-essential constant of integration. Thus $\gamma_0(\sigma)$ is known when $\Gamma_0(\sigma)$ has been determined from equation (7·18). The function $\kappa(\zeta)$, or $\kappa_0(t)$, is also known when $\Gamma_0(\sigma)$ is known, for we have, from (7·8),

$$\kappa_0(t)=-\Theta_0(t)+2i(C_1+C_2)\,m(t)\,f_0(t)+2iC_2k_0(t), \tag{7·21}$$

where
$$\Theta_0(t)=(C_1+2C_2)\int^t\Gamma_0(\sigma)\,m'(\sigma)\,d\sigma,\quad k_0(t)=\int^t f_0(\sigma)\,m'(\sigma)\,d\sigma. \tag{7·22}$$

The components t^{rs} of the stress tensor, referred to the y_i-axes, can be obtained from (5·16) when the pressure function χ and the complex displacement D have been found from (7·4) and (7·5), but it is more convenient to note that, after simplification, the following combinations of the components t^{rs} are given by

$$\begin{aligned}
t^{11}-t^{22}+2it^{12}&=2\psi^2\{\zeta\overline{\Omega}'(\overline{\zeta})+\overline{\omega}'(\overline{\zeta})-C_1\zeta^2+2i(C_1+C_2)\,\zeta\overline{f}'(\overline{\zeta})\}\\
&=2\psi^2\left[\frac{m(t)}{\overline{m}'(\overline{t})}\,\overline{\Omega}_0'(\overline{t})+\frac{\overline{\omega}_0'(\overline{t})}{\overline{m}'(\overline{t})}-C_1\{m(t)\}^2+2i(C_1+C_2)\,\frac{m(t)}{\overline{m}'(\overline{t})}\,\overline{f}_0'(\overline{t})\right],\\
t^{11}+t^{22}&=2\psi^2\{-\Omega(\zeta)-\overline{\Omega}(\overline{\zeta})+2C_1\zeta\overline{\zeta}+i(C_1+C_2)\,[2\{f(\zeta)-\overline{f}(\overline{\zeta})\}+\zeta f'(\zeta)-\overline{\zeta}\overline{f}'(\overline{\zeta})]\}\\
&=2\psi^2\Big\{-\Omega_0(t)-\overline{\Omega}_0(\overline{t})+2C_1\,m(t)\,\overline{m}(\overline{t})+i(C_1+C_2)\\
&\qquad\times\left[2\{f_0(t)-\overline{f}_0(\overline{t})\}+\frac{m(t)}{m'(t)}f_0'(t)-\frac{\overline{m}(\overline{t})}{\overline{m}'(\overline{t})}\,\overline{f}_0'(\overline{t})\right]\Big\},\\
t^{13}+it^{23}&=2(C_1+C_2)\,(\psi+i\psi^2z)\,\{\overline{f}'(\overline{\zeta})+i\zeta\}\\
&=2(C_1+C_2)\,(\psi+i\psi^2z)\,\{\overline{f}_0'(\overline{t})/\overline{m}'(\overline{t})+im(t)\},\\
t^{33}&=\psi^2\{6(C_1+C_2)\,h-\Omega(\zeta)-\overline{\Omega}(\overline{\zeta})+(C_1-2C_2)\,\zeta\overline{\zeta}+2i(C_1+C_2)\,\{f(\zeta)-\overline{f}(\overline{\zeta})\}\\
&\quad+(2C_1+C_2)\,f'(\zeta)\,\overline{f}'(\overline{\zeta})-iC_2\{\zeta f'(\zeta)-\overline{\zeta}\overline{f}'(\overline{\zeta})\}\}\\
&=\psi^2[6(C_1+C_2)\,h-\Omega_0(t)-\overline{\Omega}_0(\overline{t})+(C_1-2C_2)\,m(t)\,\overline{m}(\overline{t})\\
&\quad+2i(C_1+C_2)\,\{f_0(t)-\overline{f}_0(\overline{t})\}+(2C_1+C_2)\,f_0'(t)\,\overline{f}_0'(\overline{t})/m'(t)\,\overline{m}'(\overline{t})\\
&\quad-iC_2\{m(t)\,f_0'(t)/m'(t)-\overline{m}(\overline{t})\,\overline{f}_0'(\overline{t})/\overline{m}'(\overline{t})\}].
\end{aligned}$$

We number these expressions (7·23).

We now obtain two results which will be of use in the following work. By applying Stokes's theorem in the complex form

$$2i\iint_{R_0}\frac{\partial H}{\partial\bar{\zeta}}\,dx\,dy=\int_{C_0}H(\zeta,\bar{\zeta})\,d\zeta, \tag{7·24}$$

and by using the condition (7·2) on the function $f(\zeta)$, it can be shown that

$$\iint_{R_0}\alpha(\zeta)\,\{\bar{f}'(\bar{\zeta})+i\zeta\}\,dx\,dy=-\frac{i}{2}\int_{C_0}\alpha(\zeta)\,\{\bar{f}(\bar{\zeta})-f(\zeta)+i\zeta\bar{\zeta}\}\,d\zeta=0, \tag{7·25}$$

where $\alpha(\zeta)$ is a regular function of ζ in the domain R_0. In particular, we can show from (7·25) that

$$\iint_{R_0}i\{\zeta f'(\zeta)-\bar{\zeta}\bar{f}'(\bar{\zeta})\}\,dx\,dy=-2\iint_{R_0}f'(\zeta)\,\bar{f}'(\bar{\zeta})\,dx\,dy. \tag{7·26}$$

This result is the complex equivalent of (5·19).

Using the expression (7·4) for χ and the result (7·26), the expression (5·18) for the resultant normal force Y^3 over the end $z=l_0$ of the cylinder can be written

$$\begin{aligned}Y^3={}&\psi^2\{6h(C_1+C_2)\,A_0+(C_1-2C_2)\,I_0\}\\&+\psi^2\iint_{R_0}[-\Omega(\zeta)-\bar{\Omega}(\bar{\zeta})+2i(C_1+C_2)\,\{f(\zeta)-\bar{f}(\bar{\zeta})\}+(2C_1+3C_2)\,f'(\zeta)\,\bar{f}'(\bar{\zeta})]\,dx\,dy,\end{aligned} \tag{7·27}$$

where $$A_0=\iint_{R_0}dx\,dy,\quad I_0=\iint_{R_0}(x^2+y^2)\,dx\,dy. \tag{7·28}$$

The surface integral in (7·27) can be transformed into a line integral by applying Stokes's theorem (7·24), and we obtain

$$\begin{aligned}Y^3={}&\psi^2\{6h(C_1+C_2)\,A_0+(C_1-2C_2)\,I_0\}\\&-\frac{i}{2}\psi^2\int_{C_0}[-\bar{\zeta}\Omega(\zeta)-\bar{\Theta}(\bar{\zeta})+2i(C_1+C_2)\,\{\bar{\zeta}f(\zeta)-\bar{k}(\bar{\zeta})\}+(2C_1+3C_2)\,f'(\zeta)\,\bar{f}(\bar{\zeta})]\,d\zeta \qquad (7·29)\\={}&\psi^2\{6h(C_1+C_2)\,A_0+(C_1-2C_2)\,I_0\}\\&-\frac{i}{2}\psi^2\int_{\gamma}[-\bar{m}(\bar{t})\,\Omega_0(t)-\bar{\Theta}_0(\bar{t})+2i(C_1+C_2)\,\{\bar{m}(\bar{t})\,f_0(t)-\bar{k}_0(\bar{t})\}\\&+(2C_1+3C_2)\,f_0'(t)\,\bar{f}_0(\bar{t})/m'(t)]\,m'(t)\,dt.\end{aligned} \tag{7·30}$$

The moments m^1, m^2 given by (5·20) may be calculated when the value of χ is known, but it is perhaps simpler to note that

$$\begin{aligned}m^1-im^2={}&6i\psi^2h(C_1+C_2)\iint_{R_0}\bar{\zeta}\,dx\,dy+i\psi^2\iint_{R_0}\{\bar{\zeta}[-\Omega(\zeta)-\bar{\Omega}(\bar{\zeta})\\&+(C_1-2C_2)\,\zeta\bar{\zeta}-C_2f'(\zeta)\,\bar{f}'(\bar{\zeta})-i(C_1+2C_2)\,\{\zeta f'(\zeta)-\bar{\zeta}\bar{f}'(\bar{\zeta})\}\\&+i(C_1+C_2)\,\{3f(\zeta)-2\bar{f}(\bar{\zeta})\}]-(C_1+C_2)\,f'(\zeta)\,f(\zeta)\}\,dx\,dy.\end{aligned} \tag{7·31}$$

To obtain this expression, the results (7·25), (7·26) and the expression (7·4) for χ have been used. We could also express m^1-im^2 in terms of a line integral taken around the boundary C_0 of R_0 or around the unit circle γ in the t-plane.

In the next two sections we apply our general theory to cylinders with special cross-sections.

8. The cardioid

We consider the torsion of a beam the cross-section of which is the cardioid

$$r = 2c(1+\cos\theta),$$

where $r\,\mathrm{e}^{i\theta} = x+iy$ and c is a constant. The axis of torsion is taken to pass through the cusp of the cardioid, but the results of § 6 may be used to consider the torsion of the cylinder about a different axis.

A suitable mapping function is

$$\zeta = m(t) = c(t-1)^2, \tag{8·1}$$

and we have

$$f_0(t) = ic^2(3-4t+t^2). \tag{8·2}$$

Substituting (8·1) in the integral equation (7·18) we obtain

$$2(\sigma-1)\,\Gamma_0(\sigma)+\frac{1}{2\pi i}\int_\gamma (t-1)^2\left\{\overline{\Gamma}_0(1/t)+c^2\frac{(t-1)^4}{t^2}\right\}\frac{dt}{(t-\sigma)^2} = 0. \tag{8·3}$$

If we assume that

$$\Gamma_0(t) = \sum_{n=0}^{\infty} a_n t^n,$$

where the a_n are real constants, and use the result

$$\frac{1}{2\pi i}\int_\gamma \frac{t^m}{(t-\sigma)^2}dt = \begin{cases} m\sigma^{m-1} & (m>0), \\ 0 & (m\leqslant 0), \end{cases} \tag{8·4}$$

we find that

$$\frac{1}{2\pi i}\int_\gamma (t-1)^2\left\{\overline{\Gamma}_0(1/t)+c^2\frac{(t-1)^4}{t^2}\right\}\frac{dt}{(t-\sigma)^2} = 2a_0(\sigma-1)+a_1+c^2\{4\sigma^3-18\sigma^2+30\sigma-20\}.$$

Equation (8·3) can now be rearranged to give

$$\Gamma_0(\sigma) = -a_0-\frac{a_1}{2(\sigma-1)}+c^2\left\{\frac{2}{\sigma-1}-8+7\sigma-2\sigma^2\right\}.$$

Putting $\sigma = 0$ in this equation, before and after differentiation with respect to σ, gives two equations which enable us to determine a_0 and a_1, and we have finally

$$\Gamma_0(\sigma) = c^2\left\{-\tfrac{11}{2}+7\sigma-2\sigma^2-\frac{3}{\sigma-1}\right\}. \tag{8·5}$$

It is now a simple matter to calculate $\gamma_0(\sigma)$ from (7·20) and it is found that

$$\gamma_0(\sigma) = c^3(\sigma^2-5\sigma), \tag{8·6}$$

apart from a non-essential constant.

The complex displacement function $D(\zeta,\overline{\zeta})$ is found, by substitution in (7·5) from (7·21), (8·5) and (8·6), to be given by

$$\begin{aligned} 4(C_1+C_2)\,D = {} & c^3(C_1+2C_2)\left\{\frac{(t-1)^2}{(1-\bar{t})}(\bar{t}^2-\bar{t}+3)+14t-11t^2+6t^3-t^4-5\bar{t}+\bar{t}^2\right\} \\ & +c^3C_2\{-2\log(\bar{t}-1)+6\bar{t}-\bar{t}^2-8t+10t^2-\tfrac{16}{3}t^3+t^4\}. \end{aligned}$$

Substitution in (7·23) shows that the components t^{rs} of the stress tensor are given by

$$\begin{aligned}
t^{11}-t^{22}+2it^{12} &= 2\psi^2c^2\left\{\frac{(C_1+C_2)}{2(\bar{t}-1)}\left[\frac{(t-1)^2}{(\bar{t}-1)^2}(2+2\bar{t}-\bar{t}^2)+2\bar{t}-5\right]-C_1(t-1)^4\right. \\
&\qquad \left.+\frac{C_2}{2(\bar{t}-1)}\left[(t-1)^2\left(7-4\bar{t}+\frac{3}{(\bar{t}-1)^2}\right)+2\bar{t}-5\right]\right\}, \\
t^{11}+t^{22} &= 2\psi^2c^2\left\{2C_1(t-1)^2(\bar{t}-1)^2+C_2\left[11-7(t+\bar{t})+2(t^2+\bar{t}^2)+\frac{3}{t-1}+\frac{3}{\bar{t}-1}\right]\right. \\
&\qquad \left.+(C_1+C_2)\left[-5+4(t+\bar{t})-t^2-\bar{t}^2+\frac{3}{t-1}+\frac{3}{\bar{t}-1}\right]\right\}, \\
t^{13}+it^{23} &= 2ic(C_1+C_2)(\psi+i\psi^2z)\left\{(t-1)^2-\frac{(\bar{t}-2)}{(\bar{t}-1)}\right\}, \\
t^{33} &= 6\psi^2h(C_1+C_2)+\psi^2c^2\left\{(C_1+C_2)\left[2\frac{(t-2)(\bar{t}-2)}{(t-1)(\bar{t}-1)}+(t-1)^2(\bar{t}-1)^2\right.\right. \\
&\qquad \left.+\frac{3}{t-1}+\frac{3}{\bar{t}-1}-1+t+\bar{t}\right]+C_2\left[-\frac{(t-2)(\bar{t}-2)}{(t-1)(\bar{t}-1)}-3(t-1)^2(\bar{t}-1)^2\right. \\
&\qquad \left.\left.+\frac{3}{t-1}+\frac{3}{\bar{t}-1}+15-10(t+\bar{t})+3(t^2+\bar{t}^2)\right]\right\}.
\end{aligned}$$

In the classical solution of the torsion problem for a cardioid the stresses become infinite at the cusp. In the present solution which contains second-order terms the stresses are still infinite at the cusp, but the solution has the further drawback that the displacements there are also infinite, except when the elastic constant C_2 is zero. The stress system can, however, be maintained by a finite system of resultant forces and couples. The difficulty could be avoided by considering a slightly different transformation

$$\zeta = c(t-1-\epsilon)^2,$$

where $\epsilon > 0$, which gives a curve approximately like a cardioid if ϵ is small. The displacements and stresses are then finite everywhere and the resultant forces, obtained for the cardioid, may be considered as an approximation to the forces for this modified cross-section, when ϵ is very small.

The resultant normal force over the end $z = l_0$ of the cylinder is found from (7·30), and evaluating the line integral by the theorem of residues we obtain

$$Y^3 = \psi^2\pi c^2\{36h(C_1+C_2)+c^2(37C_1+20C_2)\}.$$

For this cross-section we have, using the notation of § 6,

$$A_{0x} = 0, \quad A_{0y} = 10\pi c^3, \quad A_0 = 6\pi c^2,$$

so that if we had taken the axis of torsion to be the line $x = -a$, $y = -b$, equation (6·12) shows that we would have then had

$$Y'^3 = \psi^2\pi c^2\{36h(C_1+C_2)+c^2(37C_1+20C_2)\}+3\psi^2\pi c^2(C_1+C_2)\{20ac+6(a^2+b^2)\}. \quad (8·7)$$

In particular, when the axis of torsion passes through the centroid $(x, y) = (\frac{5}{3}c, 0)$ of the cardioid, equation (8·7) gives

$$Y'^3 = \psi^2\pi c^2\{36h(C_1+C_2)-c^2(13C_1+30C_2)\}.$$

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9. Booth's lemniscate and epitrochoid

Two further cross-sections will be considered here, but we shall obtain only the function $\Gamma_0(t)$ in each case, since the complex displacement function and the stresses can then be obtained by straightforward calculations.

We consider first a cylinder which has a Booth's lemniscate for its cross-section, twisted about the line of centroids of the cross-sections. The mapping function

$$\zeta = m(t) = \frac{bt}{a^2+t^2} \quad (a>1,\, b>0), \tag{9·1}$$

maps the unit circle in the t-plane into a Booth's lemniscate (or the inverse of an ellipse with respect to its centre) in the ζ-plane, and the torsion function is

$$f_0(t) = \frac{ib^2(a^2-t^2)}{2(a^4-1)\,(a^2+t^2)}. \tag{9·2}$$

The integral

$$\frac{1}{2\pi i}\int_\gamma m(t)\,\bar{\Gamma}_0(1/t)\,\frac{dt}{(t-\sigma)^2} = \frac{1}{2\pi i}\int_\gamma b\bar{\Gamma}_0(1/t)\,\frac{t\,dt}{(a^2+t^2)\,(t-\sigma)^2},$$

which occurs in the integral equation (7·18), is such that the integrand is of the order of $|t|^{-3}$ for large values of t and it follows that the integral is equal to minus the sum of the residues of the integrand outside γ. Since $\bar{\Gamma}_0(1/t)$ is regular outside the unit circle, and since by symmetry we must have $\Gamma_0(t) = \Gamma_0(-t)$, we find that the integral has the value

$$-b\bar{\Gamma}_0(i/a)\,\frac{\sigma^2-a^2}{(\sigma^2+a^2)^2}.$$

Evaluating the other integral which occurs in equation (7·18) in a similar manner we obtain

$$\Gamma_0(\sigma)+\bar{\Gamma}_0(i/a) = b^2\frac{(\sigma^4-3a^2\sigma^2+3a^6\sigma^2-a^8)}{(a^4-1)^2\,(a^4-\sigma^4)}. \tag{9·3}$$

If we put $\sigma = i/a$ in this equation we have, apart from a purely imaginary constant,

$$2\Gamma_0(i/a) = -\frac{b^2(1+4a^4+a^8)}{(a^4-1)^2\,(a^4+1)},$$

and therefore, from (9·3),

$$\Gamma_0(\sigma) = -\frac{b^2(1+4a^4+a^8)}{2(a^4-1)^2\,(a^4+1)}+\frac{b^2a^2(1+a^2\sigma^2)\,\{a^2+3a^6-\sigma^2(3+a^4)\}}{(a^4-1)^2\,(a^4+1)\,(a^4-\sigma^4)}. \tag{9·4}$$

The displacements and stresses can now be determined as in the previous section.

Finally, we consider the torsion of a cylinder about the line of centroids of the cross-sections which are bounded by an epitrochoid, or a regular curvilinear polygon, so that the beam is a grooved or fluted column. The transformation

$$\zeta = m(t) = ct(1+\mu t^n) \quad (0\leqslant \mu(n+1)\leqslant 1), \tag{9·5}$$

where c and n are real and positive, maps the unit circle $|t|\leqslant 1$ upon the space inside a regular curvilinear polygon of n 'sides'. The complex torsion function is in this case

$$f_0(t) = ic^2\{\tfrac{1}{2}(1+\mu^2)+\mu t^n\}. \tag{9·6}$$

Considerations of symmetry show that $\Gamma_0(t)$ must be a function of t^n, and we therefore assume that

$$\Gamma_0(t) = \sum_{r=0}^{\infty} a_r t^{rn} \quad (a_r \text{ real}),$$

in order to evaluate the integral in equation (7·18). Employing the result (8·4) in (7·18) and dividing by $m'(\sigma)$ we obtain

$$\Gamma_0(\sigma) = -\frac{\{a_0+\mu a_1+c^2(1+2\mu^2)+\sigma^n\mu(n+1)\,[a_0+c^2(2+\mu^2)]+\sigma^{2n}\mu^2c^2(2n+1)\}}{\{1+\mu(n+1)\,\sigma^n\}}. \tag{9·7}$$

If we put $\sigma = 0$ in this equation and if we differentiate the equation with respect to σ^n and then put $\sigma = 0$, we obtain two equations which serve to determine the constants a_0 and a_1. Substituting their values in equation (9·7) we get

$$\Gamma_0(\sigma) = -c^2\{1-2n\mu^2-\mu^4(n+1)+\mu(n+1)\,\sigma^n[3-2(n+1)\,\mu^2-(n+1)\,\mu^4]$$
$$+2\mu^2(2n+1)\,\sigma^{2n}[1-\mu^2(n+1)]\}/2[1+\mu(n+1)\,\sigma^n]\,[1-\mu^2(n+1)]. \tag{9·8}$$

The working has now been carried to a stage where further results can be obtained by straightforward calculations.

TORSION SUPERPOSED UPON FINITE EXTENSION: SECOND-ORDER EFFECTS

10. STATEMENT OF THE PROBLEM AND GENERAL SOLUTION

We now consider the more general problem of determining the second-order effects accompanying the torsion of an incompressible cylinder which has previously been subjected to a finite extension λ along its length. As before the material of the cylinder is assumed to be isotropic in the unstrained state and to have the Mooney form (2·17) for the strain-energy function W'. Owing to the finite extension, however, this form for W' will not give all the possible second-order terms in torsion.

We suppose that the unstrained cylinder is of constant cross-section R_0, the generators being parallel to the x_3-axis and the ends of the cylinder being $x_3 = 0$ and $x_3 = l_0$. The cylinder is first given a uniform simple extension λ along its length so that it becomes a cylinder of constant cross-section R and length l. Since the material of which the cylinder is composed is incompressible, the point (x_1, x_2, x_3) will move to the point $(x_1/\surd\lambda, x_2/\surd\lambda, \lambda x_3)$ during this deformation and we shall write

$$x = x_1/\surd\lambda, \quad y = x_2/\surd\lambda, \quad z = \lambda x_3, \quad l = \lambda l_0. \tag{10·1}$$

We take the convected co-ordinates θ_i of a point in the elastic body to be defined by

$$\theta_1 = x+iy = (x_1+ix_2)/\surd\lambda, \quad \theta_2 = x-iy = (x_1-ix_2)/\surd\lambda, \quad \theta_3 = z, \tag{10·2}$$

and we shall write $(\zeta, \bar{\zeta}, z)$ for $(\theta_1, \theta_2, \theta_3)$. Equations will be obtained directly in terms of complex variables ζ, $\bar{\zeta}$. It follows from (10·2) that

$$g^{ik} = \begin{pmatrix} 0, & 2/\lambda, & 0 \\ 2/\lambda, & 0, & 0 \\ 0, & 0, & \lambda^2 \end{pmatrix}, \quad \surd g = \frac{i}{2}. \tag{10·3}$$

We wish to choose the final co-ordinates y_i of a point in the elastic body to be such that the cylinder is in a state of torsion about the z-axis (or x_3-axis), and such that the equilibrium of the cylinder is maintained by tractions applied to the ends of the cylinder only. For similar reasons to those given in § 5, we suppose that

$$\left.\begin{aligned} y_1 &= x-\psi yz-\tfrac{1}{2}\psi^2xz^2-\tfrac{1}{2}\psi^2hx+\psi^2U,\\ y_2 &= y+\psi xz-\tfrac{1}{2}\psi^2yz^2-\tfrac{1}{2}\psi^2hy+\psi^2V,\\ y_3 &= z+\psi\phi(x,y)+\psi^2hz,\end{aligned}\right\} \tag{10·4}$$

where the y_i-axes are coincident with the x_i-axes, and where we neglect all quantities which contain the third and higher powers of ψ, ψ being the angle of twist per unit length of the extended cylinder. The functions U, V are real single-valued functions of x and y only, ϕ is the classical torsion function for the cross-section R, and the terms containing the constant h represent a further extension $\psi^2\lambda h$ of the cylinder. Since the term in $\psi^2 W$ in (5·2) was found to be zero it is not included in (10·4). We shall see later that, when $\lambda \neq 1$, the assumptions (10·4) restrict the axis of torsion to be the line of centroids of the cross-sections of the cylinder.

In terms of ζ, $\bar\zeta$, the final co-ordinates (10·4) are

$$\left.\begin{aligned} y_1 &= \tfrac{1}{2}(\zeta+\bar\zeta)+\frac{i}{2}\psi z(\zeta-\bar\zeta)-\tfrac{1}{4}\psi^2z^2(\zeta+\bar\zeta)-\tfrac{1}{4}\psi^2h(\zeta+\bar\zeta)+\tfrac{1}{2}\psi^2(D+\bar D),\\ y_2 &= -\frac{i}{2}(\zeta-\bar\zeta)+\tfrac{1}{2}\psi z(\zeta+\bar\zeta)+\frac{i}{4}\psi^2z^2(\zeta-\bar\zeta)+\frac{i}{4}\psi^2h(\zeta-\bar\zeta)-\frac{i}{2}\psi^2(D-\bar D),\\ y_3 &= z+\tfrac{1}{2}\psi\{f(\zeta)+\bar f(\bar\zeta)\}+\psi^2hz,\end{aligned}\right\} \tag{10·5}$$

where we have put $\quad D(\zeta,\bar\zeta) = U+iV, \quad \phi(x,y) = \tfrac{1}{2}\{f(\zeta)+\bar f(\bar\zeta)\}.$

We obtain, from (10·5),

$$\left.\begin{aligned} \frac{\partial y_1}{\partial\theta_1} &= \tfrac{1}{2}+\frac{i}{2}\psi z-\tfrac{1}{4}\psi^2z^2-\tfrac{1}{4}\psi^2h+\tfrac{1}{2}\psi^2(D_\zeta+\bar D_\zeta),\\ \frac{\partial y_1}{\partial\theta_2} &= \tfrac{1}{2}-\frac{i}{2}\psi z-\tfrac{1}{4}\psi^2z^2-\tfrac{1}{4}\psi^2h+\tfrac{1}{2}\psi^2(D_{\bar\zeta}+\bar D_{\bar\zeta}),\\ \frac{\partial y_1}{\partial\theta_3} &= \frac{i}{2}\psi(\zeta-\bar\zeta)-\tfrac{1}{2}\psi^2z(\zeta+\bar\zeta),\\ \frac{\partial y_2}{\partial\theta_1} &= -\frac{i}{2}+\tfrac{1}{2}\psi z+\frac{i}{4}\psi^2z^2+\frac{i}{4}\psi^2h-\frac{i}{2}\psi^2(D_\zeta-\bar D_\zeta),\\ \frac{\partial y_2}{\partial\theta_2} &= \frac{i}{2}+\tfrac{1}{2}\psi z-\frac{i}{4}\psi^2z^2-\frac{i}{4}\psi^2h-\frac{i}{2}\psi^2(D_{\bar\zeta}-\bar D_{\bar\zeta}),\\ \frac{\partial y_2}{\partial\theta_3} &= \tfrac{1}{2}\psi(\zeta+\bar\zeta)+\frac{i}{2}\psi^2z(\zeta-\bar\zeta),\\ \frac{\partial y_3}{\partial\theta_1} &= \tfrac{1}{2}\psi f'(\zeta), \quad \frac{\partial y_3}{\partial\theta_2} = \tfrac{1}{2}\psi\bar f'(\bar\zeta), \quad \frac{\partial y_3}{\partial\theta_3} = 1+\psi^2h,\end{aligned}\right\} \tag{10·6}$$

and

$$\sqrt{G} = \frac{i}{2}\left\{1+\psi^2\left[D_\zeta+\bar D_{\bar\zeta}-\frac{i}{2}\{\zeta f'(\zeta)-\bar\zeta\bar f'(\bar\zeta)\}\right]\right\},$$

where suffixes ζ, $\bar{\zeta}$ denote differentiation with respect to these suffixes. The incompressibility condition $G = g$ is therefore

$$D_{\zeta}+\overline{D}_{\bar{\zeta}} = \frac{i}{2}\{\zeta f'(\zeta)-\bar{\zeta}\bar{f}'(\bar{\zeta})\}. \tag{10·7}$$

The covariant components G_{ik} of the metric tensor are found from (10·6) to be given by

$$\left.\begin{aligned}
G_{11} &= \psi^2[\overline{D}_{\zeta}+\tfrac{1}{4}\{f'(\zeta)\}^2],\\
G_{22} &= \psi^2[D_{\bar{\zeta}}+\tfrac{1}{4}\{\bar{f}'(\bar{\zeta})\}^2],\\
G_{33} &= 1+\psi^2(2h+\zeta\bar{\zeta}),\\
G_{12} &= \tfrac{1}{2}+\tfrac{1}{2}\psi^2\left[\tfrac{1}{2}f'(\zeta)\,\bar{f}'(\bar{\zeta})+\frac{i}{2}\{\zeta f'(\zeta)-\bar{\zeta}\bar{f}'(\bar{\zeta})\}-h\right],\\
G_{13} &= \tfrac{1}{2}\psi\{f'(\zeta)-i\bar{\zeta}\},\\
G_{23} &= \tfrac{1}{2}\psi\{\bar{f}'(\bar{\zeta})+i\zeta\}.
\end{aligned}\right\} \tag{10·8}$$

We see that the components G_{22}, G_{23} are the complex conjugates of the components G_{11}, G_{13} respectively. The contravariant components G^{ik} of the metric tensor are found to be

$$\left.\begin{aligned}
G^{11} &= \overline{G^{22}} = -\psi^2[4D_{\bar{\zeta}}-2i\zeta\bar{f}'(\bar{\zeta})+\zeta^2],\\
G^{33} &= 1+\psi^2[f'(\zeta)\,\bar{f}'(\bar{\zeta})+i\{\zeta f'(\zeta)-\bar{\zeta}\bar{f}'(\bar{\zeta})\}-2h],\\
G^{12} &= 2+\psi^2(2h+\zeta\bar{\zeta}),\\
G^{13} &= \overline{G^{23}} = -\psi\{\bar{f}'(\bar{\zeta})+i\zeta\},
\end{aligned}\right\} \tag{10·9}$$

while the first-strain invariant I_1 is given by

$$I_1 = g^{rs}G_{rs} = \frac{2}{\lambda}+\lambda^2+\psi^2\left\{\frac{1}{\lambda}\,[f'(\zeta)\,\bar{f}'(\bar{\zeta})+i\{\zeta f'(\zeta)-\bar{\zeta}\bar{f}'(\bar{\zeta})\}-2h]+\lambda^2(2h+\zeta\bar{\zeta})\right\}.$$

We can now readily show that the tensor components B^{ik} defined in § 2 are given by

$$\left.\begin{aligned}
B^{11} &= \overline{B^{22}} = -\frac{1}{\lambda^2}\psi^2[4D_{\bar{\zeta}}+\{\bar{f}'(\bar{\zeta})\}^2],\\
B^{33} &= 2\lambda+\psi^2\lambda[f'(\zeta)\bar{f}'(\bar{\zeta})+i\{\zeta f'(\zeta)-\bar{\zeta}\bar{f}'(\bar{\zeta})\}-2h],\\
B^{12} &= \frac{2}{\lambda^2}+2\lambda+\psi^2\left\{\frac{1}{\lambda^2}\,[f'(\zeta)\,\bar{f}'(\bar{\zeta})+i\{\zeta f'(\zeta)-\bar{\zeta}\bar{f}'(\bar{\zeta})\}-2h]+2\lambda(2h+\zeta\bar{\zeta})\right\},\\
B^{13} &= \overline{B^{23}} = -\psi\lambda\{\bar{f}'(\bar{\zeta})+i\zeta\}.
\end{aligned}\right\} \tag{10·10}$$

We have to choose now a suitable form for the pressure function p which occurs in the stress-strain relation (2·18). It was found in § 3 that when quantities containing powers of ψ above the first are neglected, the pressure function p is independent of ψ, and we expect the stresses τ^{11}, τ^{22}, τ^{12} to be of the order ψ^2 at least. Remembering the stress-strain relation (2·18), we assume therefore that the pressure function p has the form

$$p = -2\left\{\frac{C_1}{\lambda}+C_2\left(\lambda+\frac{1}{\lambda^2}\right)\right\}+2\psi^2\left[h\left\{\frac{C_1}{\lambda}-C_2\left(\lambda-\frac{2}{\lambda^2}\right)\right\}+\chi(\zeta,\bar{\zeta})\right],$$

where χ is a real function of ζ and $\bar{\zeta}$ only. The terms containing the constant h have been added so that the function χ is independent of h. With this value of p, the substitution of (10·3), (10·9) and (10·10) into the stress-strain relation (2·18) leads to the following values of the stress components, correct to the order ψ^2:

$$\left.\begin{aligned}
\tau^{11} &= \overline{\tau^{22}} = 2\psi^2\left[\left(\frac{C_1}{\lambda}+C_2\lambda\right)\{4D_{\bar{\zeta}}-2i\zeta\bar{f}'(\bar{\zeta})+\zeta^2\}-\frac{C_2}{\lambda^2}\{\bar{f}'(\bar{\zeta})+i\zeta\}^2\right],\\
\tau^{33} &= H+2\psi^2\left[h\left\{3\,\frac{C_1}{\lambda}-C_2\left(\lambda-\frac{4}{\lambda^2}\right)\right\}+\chi-\left(\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)\{f'(\zeta)\bar{f}'(\bar{\zeta})+i\zeta f'(\zeta)-i\bar{\zeta}\bar{f}'(\bar{\zeta})\}\right],\\
\tau^{12} &= 2\psi^2\left[2\chi+\frac{C_2}{\lambda^2}\{f'(\zeta)\bar{f}'(\bar{\zeta})+i\zeta f'(\zeta)-i\bar{\zeta}\bar{f}'(\bar{\zeta})\}-\left\{\frac{C_1}{\lambda}-C_2\left(\lambda-\frac{1}{\lambda^2}\right)\right\}\zeta\bar{\zeta}\right],\\
\tau^{13} &= \overline{\tau^{23}} = 2\psi\left(\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)\{\bar{f}'(\bar{\zeta})+i\zeta\},
\end{aligned}\right\} \tag{10·11}$$

where H is the normal surface traction on the ends of the cylinder when the torsion ψ is zero and is given by

$$H = 2\left\{C_1\left(\lambda^2-\frac{1}{\lambda}\right)+C_2\left(\lambda-\frac{1}{\lambda^2}\right)\right\}. \tag{10·12}$$

The equations of equilibrium with no body forces, obtained from (2·8), are

$$\tau^{ik}{}_{,i}+\left\{{k \atop i\;\;r}\right\}\tau^{ir} = 0$$

in this case, since we have

$$\left\{{i \atop i\;\;k}\right\} = \frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial\theta_k} = 0 \quad (k = 1, 2, 3).$$

Also the Christoffel symbols $\left\{{i \atop j\;\;k}\right\}$ of the second kind for the strained body are of the order of ψ, and remembering (10·11), we see that it is only necessary to calculate the symbols $\left\{{k \atop 13}\right\}, \left\{{k \atop 23}\right\}$ to the first order in ψ and the symbols $\left\{{k \atop 33}\right\}$ to the second order in ψ to obtain the equations of equilibrium. The first equation of equilibrium is

$$2\chi_{\bar{\zeta}}+4\left(\frac{C_1}{\lambda}+C_2\lambda\right)D_{\zeta\bar{\zeta}}-C_1\lambda^2\zeta+C_2\left[\frac{1}{\lambda^2}\{f'(\zeta)\bar{f}''(\bar{\zeta})-i\bar{\zeta}\bar{f}''(\bar{\zeta})-i\bar{f}'(\bar{\zeta})\}+2\lambda\{\zeta-i\bar{f}'(\bar{\zeta})\}\right] = 0, \tag{10·13}$$

while the second is the complex conjugate of this equation. The third equation of equilibrium is automatically satisfied. Using the incompressibility condition (10·7), it follows from (10·13) that

$$2\chi_{\zeta\bar{\zeta}}-C_1\lambda^2+C_2\left\{2\lambda+\frac{1}{\lambda^2}f''(\zeta)\bar{f}''(\bar{\zeta})\right\} = 0,$$

and we write the solution in the form

$$\begin{aligned}
2\chi = C_1\lambda^2\zeta\bar{\zeta}-C_2\left\{2\lambda\zeta\bar{\zeta}+\frac{1}{\lambda^2}f'(\zeta)\bar{f}'(\bar{\zeta})\right\}+2i\left(\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)\{f(\zeta)-\bar{f}(\bar{\zeta})\}&\\
-i\frac{C_2}{\lambda^2}\{\zeta f'(\zeta)-\bar{\zeta}\bar{f}'(\bar{\zeta})\}-\Omega(\zeta)-\bar{\Omega}(\bar{\zeta}),&
\end{aligned} \tag{10·14}$$

in order to simplify the boundary conditions. The function $\Omega(\zeta)$ is a regular function of ζ in the domain R, and, although it is undetermined to the extent of a purely imaginary constant, this constant does not affect the values of the stress components. The substitution of (10·14) into (10·13) gives

$$4\left(\frac{C_1}{\lambda}+C_2\lambda\right)D_{\zeta\bar{\zeta}}=\bar{\Omega}'(\bar{\zeta})+2i\left\{\frac{C_1}{\lambda}+C_2\left(\lambda+\frac{1}{\lambda^2}\right)\right\}\bar{f}'(\bar{\zeta}),$$

so that
$$4\left(\frac{C_1}{\lambda}+C_2\lambda\right)D=2i\left\{\frac{C_1}{\lambda}+C_2\left(\lambda+\frac{1}{\lambda^2}\right)\right\}\zeta\bar{f}(\bar{\zeta})+\zeta\bar{\Omega}(\bar{\zeta})+\bar{\omega}(\bar{\zeta})+\kappa(\zeta)+\frac{C_2}{\lambda^2}\bar{g}(\bar{\zeta}), \qquad (10\cdot15)$$

where $\omega(\zeta)$ and $\kappa(\zeta)$ are regular functions of ζ in the domain R. As before, the function $g(\zeta)$ is given by (7·6), and the term containing this function has been added to simplify the boundary conditions.

The function $\kappa(\zeta)$ can be expressed in terms of the functions $\Omega(\zeta)$, $\omega(\zeta)$ by means of the incompressibility condition (10·7). For, on substituting (10·15) into this condition, we obtain

$$2i\left\{\frac{C_1}{\lambda}+C_2\left(\lambda+\frac{1}{\lambda^2}\right)\right\}\{\bar{f}(\bar{\zeta})-f(\zeta)\}+\bar{\Omega}(\bar{\zeta})+\Omega(\zeta)+\kappa'(\zeta)+\bar{\kappa}'(\bar{\zeta})=2i\left(\frac{C_1}{\lambda}+C_2\lambda\right)\{\zeta f'(\zeta)-\bar{\zeta}\bar{f}'(\bar{\zeta})\},$$

and this implies that

$$\kappa(\zeta)=-\Theta(\zeta)+2i\left(\frac{C_1}{\lambda}+C_2\lambda\right)\zeta f(\zeta)+2i\frac{C_2}{\lambda^2}k(\zeta)+i\beta\zeta, \qquad (10\cdot16)$$

where $\Theta(\zeta)$ and $k(\zeta)$ are given by (7·9). The term $i\beta\zeta$, where β is a real constant, may be omitted from the expression for $\kappa(\zeta)$ as the displacements arising from this term do not contribute to the stresses.

The expressions (10·14) and (10·15) for χ and D enable us to write the stress components (10·11) in the form

$$\left.\begin{aligned}\tau^{11}&=\overline{\tau^{22}}=2\psi^2\left[\zeta\bar{\Omega}'(\bar{\zeta})+\bar{\omega}'(\bar{\zeta})+\left\{\frac{C_1}{\lambda}+C_2\left(\lambda+\frac{1}{\lambda^2}\right)\zeta^2\right\}\right],\\ \tau^{33}&=H+\psi^2\left[2h\left\{3\frac{C_1}{\lambda}-C_2\left(\lambda-\frac{4}{\lambda^2}\right)\right\}-\Omega(\zeta)-\bar{\Omega}(\bar{\zeta})+(C_1\lambda^2-2C_2\lambda)\zeta\bar{\zeta}\right.\\ &\qquad\left.+2i\left(\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)\{f(\zeta)-\bar{f}(\bar{\zeta})\}-\left(\frac{2C_1}{\lambda}+\frac{3C_2}{\lambda^2}\right)\{f'(\zeta)\bar{f}'(\bar{\zeta})+i\zeta f'(\zeta)-i\bar{\zeta}\bar{f}'(\bar{\zeta})\}\right],\\ \tau^{12}&=2\psi^2\left[-\Omega(\zeta)-\bar{\Omega}(\bar{\zeta})+2i\left(\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)\{f(\zeta)-\bar{f}(\bar{\zeta})\}+\left\{C_1\left(\lambda^2-\frac{1}{\lambda}\right)-C_2\left(\lambda+\frac{1}{\lambda^2}\right)\right\}\zeta\bar{\zeta}\right],\\ \tau^{13}&=\overline{\tau^{23}}=2\psi\left(\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)\{\bar{f}'(\bar{\zeta})+i\zeta\}.\end{aligned}\right\} \qquad (10\cdot17)$$

The boundary conditions on the cylindrical surface will be considered next. We suppose that the curved surface of the cylinder in the strained state is the surface

$$G(\zeta,\bar{\zeta})=0, \qquad (10\cdot18)$$

where
$$F(x_1/\surd\lambda,\,x_2/\surd\lambda)\equiv G\{(x_1+ix_2)/\surd\lambda,\,(x_1-ix_2)/\surd\lambda\}=0 \qquad (10\cdot19)$$

is the curved surface of the cylinder in the unstrained state, and G is a real function of ζ and $\bar{\zeta}$. The equation (10·18) is to be interpreted as the parametric equation of the surface, and the co-ordinates y_i of a point on the strained surface are given in terms of the parameters ζ, $\bar{\zeta}$ by means of the equations (10·5). As in § 3, it can be shown that the covariant components n_i, referred to the base vectors $\mathbf{E}^i$, of the unit normal $\mathbf{n}$ to the surface (10·18) are such that

$$n_1:n_2:n_3=G_{\zeta}:G_{\bar{\zeta}}:0.$$

The torsion function $f(\zeta)$ satisfies the boundary condition

$$G_\zeta\{\bar{f}'(\bar\zeta)+i\zeta\}+G_{\bar\zeta}\{f'(\zeta)-i\bar\zeta\}=0 \quad\text{or}\quad f(\zeta)-\bar{f}(\bar\zeta)=i\zeta\bar\zeta \tag{10·20}$$

on the surface (10·18), and, using (10·20), the boundary conditions (2·11) on the curved surface, which we suppose to be free from traction, reduce to the condition

$$G_\zeta\left[\zeta\bar\Omega'(\bar\zeta)+\bar\omega'(\bar\zeta)+\left\{\frac{C_1}{\lambda}+C_2\left(\lambda+\frac{1}{\lambda^2}\right)\right\}\zeta^2\right]$$
$$+G_{\bar\zeta}\left[-\Omega(\zeta)-\bar\Omega(\bar\zeta)+\left\{C_1\left(\lambda^2-\frac{3}{\lambda}\right)-C_2\left(\lambda+\frac{3}{\lambda^2}\right)\right\}\zeta\bar\zeta\right]=0 \tag{10·21}$$

on $G(\zeta,\bar\zeta)=0$. The second boundary condition is the complex conjugate of this one, and the third condition is automatically satisfied. To remove the elastic constants C_1, C_2 and the extension ratio λ from the boundary condition (10·21) we write

$$\left.\begin{aligned}\Omega(\zeta)&=\left\{\frac{C_1}{\lambda}+C_2\left(\lambda+\frac{1}{\lambda^2}\right)\right\}\Gamma(\zeta)+\left\{C_1\left(\lambda^2-\frac{1}{\lambda}\right)+C_2\left(\lambda-\frac{1}{\lambda^2}\right)\right\}\Delta(\zeta),\\ \omega(\zeta)&=\left\{\frac{C_1}{\lambda}+C_2\left(\lambda+\frac{1}{\lambda^2}\right)\right\}\gamma(\zeta)+\left\{C_1\left(\lambda^2-\frac{1}{\lambda}\right)+C_2\left(\lambda-\frac{1}{\lambda^2}\right)\delta(\zeta),\right.\end{aligned}\right\} \tag{10·22}$$

and then the four canonical functions $\Gamma(\zeta)$, $\gamma(\zeta)$, $\Delta(\zeta)$, $\delta(\zeta)$ are such that

$$\left.\begin{aligned}G_\zeta[\zeta\bar\Gamma'(\bar\zeta)+\bar\gamma'(\bar\zeta)+\zeta^2]-G_{\bar\zeta}[\Gamma(\zeta)+\bar\Gamma(\bar\zeta)+2\zeta\bar\zeta]&=0,\\ G_\zeta[\zeta\bar\Delta'(\bar\zeta)+\bar\delta'(\bar\zeta)]-G_{\bar\zeta}[\Delta(\zeta)+\bar\Delta(\bar\zeta)-\zeta\bar\zeta]&=0\end{aligned}\right\} \tag{10·23}$$

on $G(\zeta,\bar\zeta)=0$.

The first condition of (10·23) is the same as the condition (7·12) imposed upon the functions $\Gamma(\zeta)$, $\gamma(\zeta)$ of § 7, and the method of solution used in § 7 is applicable here to determine the functions $\Gamma(\zeta)$, $\gamma(\zeta)$. Thus, if we suppose that the transform $\zeta=m(t)$ maps the domain R upon the interior of the unit circle γ in the t-plane then, under suitable conditions, $\Gamma(\zeta)$ and $\gamma(\zeta)$ are given by (7·18) and (7·20), where we again use the notation (7·15).

With (7·3), the second boundary condition of (10·23) can be written

$$d\left\{\zeta\bar\Delta(\bar\zeta)+\bar\delta(\bar\zeta)+\int^\zeta\Delta(\sigma)\,d\sigma\right\}-\zeta\bar\zeta\,d\zeta=0$$

on $G(\zeta,\bar\zeta)=0$. Integrating this condition we obtain

$$\zeta\bar\Delta(\bar\zeta)+\bar\delta(\bar\zeta)+\int^\zeta\Delta(\sigma)\,d\sigma=\int^\zeta\zeta\bar\zeta\,d\zeta \tag{10·23a}$$

on $G(\zeta,\bar\zeta)=0$, where the path of integration of the line integral on the right-hand side lies on the boundary C of R. The functions $\Delta(\zeta)$, $\delta(\zeta)$ are regular in R, and (10·23*a*) shows that we must have

$$\int_C\zeta\bar\zeta\,d\zeta=0$$

if $\Delta(\zeta)$, $\delta(\zeta)$ exist. Applying Stokes's theorem (7·24), this condition is equivalent to

$$\iint_R\zeta\,dx\,dy=0,$$

and we see that the z-axis must pass through the centroid of the cross-section R of the cylinder.

When this condition is satisfied, i.e. when the axis of torsion is the line of centroids of the cross-sections, it can be shown that, under suitable conditions, the functions $\Delta(\zeta)$, $\delta(\zeta)$ are given by

$$\left.\begin{aligned} m'(\sigma)\,\Delta_0(\sigma)+\frac{1}{2\pi i}\int_\gamma m(t)\overline{\Delta}_0(1/t)\,\frac{dt}{(t-\sigma)^2} &= \frac{1}{2\pi i}\int_\gamma m(t)\overline{m}(1/t)\,\frac{m'(t)\,dt}{(t-\sigma)},\\ \delta_0'(\sigma) &= -\frac{1}{2\pi i}\int_\gamma m(t)\overline{m}(1/t)\,\frac{\overline{m}'(1/t)\,dt}{t^2(t-\sigma)}-\frac{1}{2\pi i}\int_\gamma \overline{m}(1/t)\,\Delta_0(t)\,\frac{dt}{(t-\sigma)^2}. \end{aligned}\right\} \qquad (10{\cdot}24)$$

When the functions $\Gamma(\zeta)$, $\gamma(\zeta)$, $\Delta(\zeta)$, $\delta(\zeta)$ have been determined from the boundary conditions (10·23), or alternatively from the integral equations (7·18), (7·20) and (10·24), the function $\kappa(\zeta)$ can be found by substitution in (10·16) and the displacement function $D(\zeta,\overline{\zeta})$ can then be obtained by substitution in (10·15).

The stress components τ^{rs} referred to the convected co-ordinates θ_i are given by (10·17), but it is perhaps more convenient to know the stress components referred to co-ordinate systems in the strained body. To find these, we write

$$Z_1 = y_1+iy_2 = Z, \quad Z_2 = y_1-iy_2, = \overline{Z}, \quad Z_3 = y_3, \qquad (10{\cdot}25)$$

and we let T^{rs} and t^{rs} be the components of the stress tensor referred to the Z_i-axes and the y_i-axes respectively. It can easily be shown that

$$\begin{aligned} T^{11} &= \overline{T^{22}} = t^{11}-t^{22}+2it^{12}, & T^{12} &= t^{11}+t^{22},\\ T^{13} &= \overline{T^{23}} = t^{13}+it^{23}, & T^{33} &= t^{33}. \end{aligned}$$

From (10·5) and (10·25) we have

$$\left.\begin{aligned} Z &= \zeta+i\psi z\zeta-\tfrac{1}{2}\psi^2z^2\zeta-\tfrac{1}{2}\psi^2h\zeta+\psi^2D,\\ Z_3 &= z+\tfrac{1}{2}\psi\{f(\zeta)+\overline{f}(\overline{\zeta})\}+\psi^2hz, \end{aligned}\right\} \qquad (10{\cdot}26)$$

and it follows that

$$\frac{\partial Z_i}{\partial\theta_k} = \begin{pmatrix} 1+i\psi z-\tfrac{1}{2}\psi^2z^2-\tfrac{1}{2}\psi^2h+\psi^2D_\zeta, & \psi^2D_{\overline{\zeta}}, & i\psi\zeta-\psi^2\zeta z\\ \psi^2\overline{D}_\zeta, & 1-i\psi z-\tfrac{1}{2}\psi^2z^2-\tfrac{1}{2}\psi^2h+\psi^2\overline{D}_{\overline{\zeta}}, & -i\psi\overline{\zeta}-\psi^2\overline{\zeta}z\\ \tfrac{1}{2}\psi f'(\zeta), & \tfrac{1}{2}\psi\overline{f}'(\overline{\zeta}), & 1+\psi^2h \end{pmatrix}. \qquad (10{\cdot}27)$$

Because of the tensor character of the stress components τ^{rs}, T^{rs}, we have

$$T^{rs} = \tau^{ik}\frac{\partial Z_r}{\partial\theta_i}\frac{\partial Z_s}{\partial\theta_k},$$

and (10·17), (10·27) give

$$\left.\begin{aligned} T^{11} &= \overline{T^{22}} = 2\psi^2\left\{\zeta\overline{\Omega}'(\overline{\zeta})+\overline{\omega}'(\overline{\zeta})-C_1\lambda^2\zeta^2+2i\left(\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)\zeta\overline{f}'(\overline{\zeta})\right\},\\ T^{12} &= 2\psi^2\left[-\Omega(\zeta)-\overline{\Omega}(\overline{\zeta})+2C_1\lambda^2\zeta\overline{\zeta}+i\left(\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)\{2f(\zeta)-2\overline{f}(\overline{\zeta})+\zeta f'(\zeta)-\overline{\zeta}\overline{f}'(\overline{\zeta})\}\right],\\ T^{13} &= \overline{T^{23}} = (\psi+i\psi^2z)\left[2\left(\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)\{\overline{f}'(\overline{\zeta})+i\zeta\}+iH\zeta\right],\\ T^{33} &= H+2\psi^2h\left\{C_1\left(2\lambda^2+\frac{1}{\lambda}\right)+C_2\left(\lambda+\frac{2}{\lambda^2}\right)\right\}+\psi^2\left[-\Omega(\zeta)-\overline{\Omega}(\overline{\zeta})+(C_1\lambda^2-2C_2\lambda)\,\zeta\overline{\zeta}\right.\\ &\quad\left.+2i\left(\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)\{f(\zeta)-\overline{f}(\overline{\zeta})\}+\left(2\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)f'(\zeta)\,\overline{f}'(\overline{\zeta})-i\frac{C_2}{\lambda^2}\{\zeta f'(\zeta)-\overline{\zeta}\overline{f}'(\overline{\zeta})\}\right]. \end{aligned}\right\} \qquad (10{\cdot}28)$$

We notice that the first two terms in the expression for T^{33} can be written, to our order of approximation,

$$2C_1\left\{\lambda^2(1+\psi^2h)^2-\frac{1}{\lambda(1+\psi^2h)}\right\}+2C_2\left\{\lambda(1+\psi^2h)-\frac{1}{\lambda^2(1+\psi^2h)^2}\right\}.$$

We consider now the surface tractions on the end $z = l = \lambda l_0$ of the cylinder. The unit normal vector to the surface $z = l$ is the vector $\mathbf{n} = n_3\mathbf{E}^3$, where

$$n_3 = \frac{1}{\sqrt{G^{33}}} = 1+O(\psi^2).$$

The components of surface traction (2·11), referred to the θ_i-axes, are therefore

$$P^k = (\tau^{13}, \tau^{23}, \tau^{33}/\sqrt{G^{33}})_{z=l},$$

to our order of approximation, and if we write Q^k for the components of surface traction referred to the Z_i-axes, then

$$\begin{aligned} Q^k &= \left(P^r\frac{\partial Z_k}{\partial \theta_r}\right)_{z=l} \\ &= \left\{(\tau^{13}+i\psi\zeta\tau^{33})(1+i\psi z),(\tau^{23}-i\psi\bar{\zeta}\tau^{33})(1-i\psi z),\tfrac{1}{2}\psi\tau^{13}f'(\zeta)+\tfrac{1}{2}\psi\tau^{23}\bar{f}'(\bar{\zeta})+\frac{\tau^{33}}{\sqrt{G^{33}}}(1+\psi^2h)\right\}_{z=l}. \end{aligned}$$

The element of area dS on the surface $z = l$ in the strained state is given by

$$dS = \sqrt{(GG^{33})}\,d\theta^1\,d\theta^2 = \frac{i}{2}\sqrt{G^{33}}\,d\zeta\,d\bar{\zeta} = \sqrt{G^{33}}\,dx\,dy.$$

Hence, to our order of approximation, we have

$$\begin{aligned} Q^k\,dS = dx\,dy\Big\{T^{13},\ T^{23},\ H+2\psi^2h\left[C_1\left(\lambda^2+\frac{2}{\lambda}\right)+3\frac{C_2}{\lambda^2}\right] \\ +\psi^2\Big[-\Omega(\zeta)-\bar{\Omega}(\bar{\zeta})+(C_1\lambda^2-2C_2\lambda)\,\zeta\bar{\zeta}+2i\left(\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)\{f(\zeta)-\bar{f}(\bar{\zeta})\} \\ -\frac{C_2}{\lambda^2}f'(\zeta)\,\bar{f}'(\bar{\zeta})-i\left(\frac{C_1}{\lambda}+2\frac{C_2}{\lambda^2}\right)\{\zeta f'(\zeta)-\bar{\zeta}\bar{f}'(\bar{\zeta})\}\Big]\Big\}_{z=l}. \end{aligned}$$

The components of the resultant force $\mathbf{N}$ over the end of the cylinder may be referred to the Z_i-axes or to the y_i-axes, and we write

$$\mathbf{N} = N^r\mathbf{I}_r = Y^r\mathbf{i}_r,$$

where $\mathbf{I}_r$ are covariant base vectors for the Z_i co-ordinate system and $\mathbf{i}_r$ are unit vectors along the y_r-axes. Then we have

$$\begin{aligned} N^1 = Y^1+iY^2 &= \iint_R Q^1\,dS \\ &= i\psi(1+i\psi l)\,H\iint_R \zeta\,dx\,dy = 0. \end{aligned} \tag{10·29}$$

We see that there will be no resultant transverse force since the axis of torsion passes through the centroid of the cross-section. The third component of resultant force is given by

$$\begin{aligned} N^3 = Y^3 &= \iint_R Q^3\,dS \\ &= \left[H+2\psi^2h\left\{C_1\left(\lambda^2+\frac{2}{\lambda}\right)+3\frac{C_2}{\lambda^2}\right\}\right]A+\psi^2(C_1\lambda^2-2C_2\lambda)\,I \\ &\quad +\psi^2\iint_R\left[-\Omega(\zeta)-\bar{\Omega}(\bar{\zeta})+2i\left(\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)\{f(\zeta)-\bar{f}(\bar{\zeta})\}+\left(2\frac{C_1}{\lambda}+3\frac{C_2}{\lambda^2}\right)f'(\zeta)\,\bar{f}'(\bar{\zeta})\right]dx\,dy, \end{aligned} \tag{10·30}$$

where we have used the result (7·26) and where

$$A = \iint_R dx\,dy, \quad I = \iint_R (x^2+y^2)\,dx\,dy.$$

The surface integral in the expression (10·30) can be transformed into a line integral around the boundary C of the domain R by using Stokes's theorem (7·24), and we obtain

$$\begin{aligned} Y^3 = {} & \left[H+2\psi^2 h\left\{C_1\left(\lambda^2+\frac{2}{\lambda}\right)+3\frac{C_2}{\lambda^2}\right\}\right]A+\psi^2(C_1\lambda^2-2C_2\lambda)\,I \\ & -\frac{i}{2}\psi^2\int_C\left[-\bar{\zeta}\Omega(\zeta)-\bar{\Theta}(\bar{\zeta})+2i\left(\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)\{\bar{\zeta}f(\zeta)-\bar{k}(\bar{\zeta})\}+\left(2\frac{C_1}{\lambda}+3\frac{C_2}{\lambda^2}\right)f'(\zeta)\,\bar{f}(\bar{\zeta})\right]d\zeta. \end{aligned} \tag{10·31}$$

It is also possible to express Y^3 in terms of a line integral around the unit circle in the t-plane by using the transformation $\zeta = m(t)$.

The moment per unit area of the traction on $z = l$ about the point $y_i = (0, 0, l)$ is $\mathbf{R}\times\mathbf{P}$, where $\mathbf{R}$ is the vector from the point $y_i = (0, 0, l)$ to a current point on $z = l$ and $\mathbf{P}$ is the surface force at that point. We have

$$\mathbf{R} = \{Z_1\mathbf{I}_1+Z_2\mathbf{I}_2+(Z_3-l)\,\mathbf{I}_3\}_{z=l}, \quad \mathbf{P} = Q^k\mathbf{I}_k,$$

and therefore

$$\mathbf{R}\times\mathbf{P} = \frac{i}{2}[\{Z_2Q^3-(Z_3-l)\,Q^2\}\,\mathbf{I}^1+\{(Z_3-l)\,Q^1-Z_1Q^3\}\,\mathbf{I}^2+\{Z_1Q^2-Z_2Q^1\}\,\mathbf{I}^3]_{z=l}, \tag{10·32}$$

since

$$\mathbf{I}_r\times\mathbf{I}_s = \frac{i}{2}e_{rst}\mathbf{I}^t,$$

where $e_{rst} = \pm 1$ according as r, s, t is an even or odd permutation of the numbers 1, 2, 3, and is 0 otherwise. The resultant moment $\mathbf{M}$ of the surface tractions on the end $z = l$ is obtained by integrating (10·32) over the end of the cylinder, and, if we write

$$\mathbf{M} = M_r\mathbf{I}^r = m_r\mathbf{i}^r,$$

we have, using the results (7·25), (7·26) and (10·29),

$$\begin{aligned} M_1 = \tfrac{1}{2}(m_1-im_2) & = \frac{i}{2}\iint_R\{Z_2Q^3-(Z_3-l)\,Q^2\}_{z=l}\,dS \\ & = \frac{i}{2}H\psi^2\iint_R\bar{D}\,dx\,dy \\ & \quad +\frac{i}{2}\psi^2\iint_R\left\{\bar{\zeta}\left[-\Omega(\zeta)-\bar{\Omega}(\bar{\zeta})+(C_1\lambda^2-2C_2\lambda)\,\zeta\bar{\zeta}-\frac{C_2}{\lambda^2}f'(\zeta)\,\bar{f}'(\bar{\zeta})\right.\right. \\ & \quad -i\left(\frac{C_1}{\lambda}+2\frac{C_2}{\lambda^2}\right)\{\zeta f'(\zeta)-\bar{\zeta}\bar{f}'(\bar{\zeta})\}+i\left\{C_1\left(\lambda^2+\frac{2}{\lambda}\right)+C_2\left(\lambda+\frac{2}{\lambda^2}\right)\right\}f(\zeta) \\ & \quad \left.\left.+i\left\{C_1\left(\lambda^2-\frac{3}{\lambda}\right)+C_2\left(\lambda-\frac{3}{\lambda^2}\right)\right\}\bar{f}(\bar{\zeta})\right]-\left(\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)f(\zeta)\,f'(\zeta)\right\}dx\,dy, \end{aligned} \tag{10·33}$$

$$\begin{aligned} M_3 = m_3 & = \frac{i}{2}\iint_R\{Z_1Q^2-Z_2Q^1\}_{z=l}\,dS \\ & = 2\psi\left(\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)\iint_R\left[\frac{i}{2}\{\zeta f'(\zeta)-\bar{\zeta}\bar{f}'(\bar{\zeta})\}+\zeta\bar{\zeta}\right]dx\,dy+\psi H\iint_R\zeta\bar{\zeta}\,dx\,dy \\ & = 2\psi\left(\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)S+\psi HI, \end{aligned} \tag{10·34}$$

where S is given by (3·24). Formula (10·34) for m_3 can also be deduced, as a special case, from (3·23).

The theory developed in this section can be modified to include cross-sections which are not simply connected. For multiply-connected regions, the boundary condition on the functions $\Omega(\zeta)$, $\omega(\zeta)$ will differ from the condition (10·21), since, in general, the complex torsion function $f(\zeta)$ will satisfy the condition

$$f(\zeta)-\bar{f}(\bar{\zeta}) = i\zeta\bar{\zeta}$$

on only one of the closed curves bounding the cross-section. Also the solution by conformal transformation on to the unit circle will not apply in this case.

11. Elliptical cylinder

The theory of the preceding section will be applied here to consider the torsion of an elliptical cylinder about its axis, but the solution is obtained without using the integral equations. The unstrained cross-section is the ellipse

$$\frac{x_1^2}{a_0^2}+\frac{x_2^2}{b_0^2} = 1,$$

with semi-axes a_0 and b_0, and after the finite extension λ along the length of the cylinder has been imposed, the cross-section becomes the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1, \tag{11·1}$$

where $$a = a_0/\sqrt{\lambda}, \quad b = b_0/\sqrt{\lambda}.$$

Equation (11·1) is the parametric equation of the curved surface of the cylinder in the strained state and the equation can be written

$$G(\zeta,\bar{\zeta}) = \zeta\bar{\zeta}+\tfrac{1}{2}\alpha(\zeta^2+\bar{\zeta}^2)-\frac{(\alpha^2-1)}{2\alpha}(a^2-b^2) = 0, \tag{11·2}$$

where $$\alpha = -\frac{a^2-b^2}{a^2+b^2} = -\frac{a_0^2-b_0^2}{a_0^2+b_0^2}.$$

The complex torsion function $f(\zeta)$ is given by

$$f(\zeta) = -\tfrac{1}{2}i\alpha\zeta^2+i\frac{(\alpha^2-1)}{4\alpha}(a^2-b^2). \tag{11·3}$$

From considerations of symmetry, the functions $\Gamma(\zeta)$, $\gamma'(\zeta)$ defined by (10·22) must be even functions of ζ, and we assume therefore that

$$\Gamma(\zeta) = A\zeta^2+B, \quad \gamma'(\zeta) = L\zeta^2+M, \tag{11·4}$$

where A, B, L, M are real constants. Substituting (11·2) and (11·4) in the first of the boundary conditions (10·23), we obtain

$$(\bar{\zeta}+\alpha\zeta)\,[2A\zeta\bar{\zeta}+L\bar{\zeta}^2+M+\zeta^2]-(\zeta+\alpha\bar{\zeta})\,[A\zeta^2+A\bar{\zeta}^2+2B+2\zeta\bar{\zeta}] = 0 \tag{11·5}$$

on the surface (11·2). The condition (11·5) contains terms in ζ, $\bar{\zeta}$, $\zeta\bar{\zeta}^2$, $\zeta^2\bar{\zeta}$, ζ^3 and $\bar{\zeta}^3$, but the terms involving ζ^3 and $\bar{\zeta}^3$ can be removed by using the equation of the boundary, $G(\zeta,\bar{\zeta}) = 0$. Equating the coefficients of the terms ζ, $\bar{\zeta}$, $\zeta\bar{\zeta}^2$, $\zeta^2\bar{\zeta}$ in the resulting equation to zero, we get

$$\left.\begin{aligned} M-2\alpha B+(L-\alpha A)\frac{(\alpha^2-1)}{\alpha^2}(a^2-b^2) &= 0,\\ \alpha M-2B+(\alpha-A)\frac{(\alpha^2-1)}{\alpha^2}(a^2-b^2) &= 0,\\ 2A\left(\alpha+\frac{1}{\alpha}\right)-L-3 &= 0,\\ 4A+L\left(\alpha-\frac{2}{\alpha}\right)-3\alpha &= 0,\end{aligned}\right\} \tag{11·6}$$

and therefore

$$\left.\begin{aligned} A &= \frac{3\alpha}{\alpha^2+2}, \quad & B &= -\frac{(a^2-b^2)(\alpha^2-1)}{2\alpha(\alpha^2+2)},\\ L &= \frac{3\alpha^2}{\alpha^2+2}, \quad & M &= -\frac{(a^2-b^2)(\alpha^2-1)}{(\alpha^2+2)}.\end{aligned}\right\} \tag{11·7}$$

Substituting these values in equations (11·4) we find that

$$\Gamma(\zeta) = \frac{3\alpha}{\alpha^2+2}\zeta^2-\frac{(a^2-b^2)(\alpha^2-1)}{2\alpha(\alpha^2+2)}, \quad \gamma(\zeta) = \frac{\alpha^2}{(\alpha^2+2)}\zeta^3-\frac{(a^2-b^2)(\alpha^2-1)}{(\alpha^2+2)}\zeta.$$

In the same way we obtain

$$\Delta(\zeta) = -\frac{\alpha}{\alpha^2+2}\zeta^2-\frac{(a^2-b^2)}{2\alpha(\alpha^2+2)}, \quad \delta(\zeta) = -\frac{\alpha^2}{3(\alpha^2+2)}\zeta^3-\frac{(a^2-b^2)}{(\alpha^2+2)}\zeta.$$

The functions $\Omega(\zeta)$, $\omega(\zeta)$ can now be found by substituting in (10·22).

The complex displacement function is found from (10·15), (10·16) and (11·3) to be given by

$$\begin{aligned} 4(\alpha^2+2)\left(\frac{C_1}{\lambda}+C_2\lambda\right)D(\zeta,\bar{\zeta}) = {} & \tfrac{1}{3}\alpha\zeta^3\left\{\alpha^2\left[3\frac{C_1}{\lambda}+C_2\left(3\lambda+\frac{1}{\lambda^2}\right)\right]+C_1\left(\lambda^2+\frac{2}{\lambda}\right)+2C_2\left(2\lambda-\frac{1}{\lambda^2}\right)\right\}\\ & -\tfrac{1}{3}\alpha^2\bar{\zeta}^3\left\{\alpha^2\frac{C_2}{\lambda^2}+C_1\left(\lambda^2-\frac{4}{\lambda}\right)-2C_2\left(\lambda+\frac{1}{\lambda^2}\right)\right\}\\ & -\{\alpha\zeta\bar{\zeta}^2+(a^2-b^2)\bar{\zeta}\}\left\{\alpha^2\left[\frac{C_1}{\lambda}+C_2\left(\lambda+\frac{1}{\lambda^2}\right)\right]+C_1\left(\lambda^2-\frac{2}{\lambda}\right)-2\frac{C_2}{\lambda^2}\right\},\end{aligned}$$

while the stress components T^{rs} are given by

$$\begin{aligned} T^{11} = \overline{T^{22}} = \frac{2\psi^2}{\alpha^2+2}\Bigg[& -2\alpha\zeta\bar{\zeta}\left\{\alpha^2\left(\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)+C_1\left(\lambda^2-\frac{2}{\lambda}\right)-2C_2\left(\lambda+\frac{1}{\lambda^2}\right)\right\}-(\alpha^2+2)C_1\lambda^2\zeta^2\\ & -\alpha^2\bar{\zeta}^2\left\{C_1\left(\lambda^2-\frac{4}{\lambda}\right)-2C_2\left(\lambda+\frac{2}{\lambda^2}\right)\right\}-(a^2-b^2)\left\{\alpha^2\left[\frac{C_1}{\lambda}+C_2\left(\lambda+\frac{1}{\lambda^2}\right)\right]+C_1\left(\lambda^2-\frac{2}{\lambda}\right)-2\frac{C_2}{\lambda^2}\right\}\Bigg],\end{aligned}$$

$$\begin{aligned} T^{12} = \frac{2\psi^2}{\alpha^2+2}\Bigg[& 2(\alpha^2+2)C_1\lambda^2\zeta\bar{\zeta}+\alpha(\zeta^2+\bar{\zeta}^2)\left\{2\alpha^2\left(\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)+C_1\lambda^2-2C_2\lambda\right\}\\ & +\frac{(a^2-b^2)}{\alpha}\left\{-\alpha^4\left(\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)+\alpha^2C_2\lambda+C_1\lambda^2\right\}\Bigg],\end{aligned}$$

$$T^{13} = \overline{T^{23}} = 2i\psi(1+i\psi z)\left\{\alpha\bar{\zeta}\left(\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)+\zeta(C_1\lambda^2+C_2\lambda)\right\},$$

$$T^{33} = H+2\psi^2h\left\{C_1\left(2\lambda^2+\frac{1}{\lambda}\right)+C_2\left(\lambda+\frac{2}{\lambda^2}\right)\right\}+\psi^2\left[\zeta\bar{\zeta}\left\{\alpha^2\left(2\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)+C_1\lambda^2-2C_2\lambda\right\}\right.$$
$$+\frac{\alpha(\zeta^2+\bar{\zeta}^2)}{(\alpha^2+2)}\left\{\alpha^2\frac{C_1}{\lambda}+C_1\left(\lambda^2-\frac{2}{\lambda}\right)-2C_2\left(\lambda+\frac{2}{\lambda^2}\right)\right\}$$
$$\left.+\frac{(a^2-b^2)}{\alpha(\alpha^2+2)}\left\{-\alpha^4\left(\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)+\alpha^2C_2\lambda+C_1\lambda^2\right\}\right].$$

The resultant normal force Y^3 over the end $z = l_0$ of the cylinder can be found from (10·30), and, omitting the details of the calculation, we have

$$Y^3 = \pi ab\left[H+2\psi^2h\left\{C_1\left(\lambda^2+\frac{2}{\lambda}\right)+3\frac{C_2}{\lambda^2}\right\}\right]$$
$$+\psi^2\frac{\pi ab}{4}\frac{(a^2+b^2)}{(\alpha^2+2)}\left[\alpha^4\left(4\frac{C_1}{\lambda}+5\frac{C_2}{\lambda^2}\right)-\alpha^2\left\{C_1\left(\lambda^2-\frac{8}{\lambda}\right)+2C_2\left(\lambda-\frac{5}{\lambda^2}\right)\right\}-2C_1\lambda^2-4C_2\lambda\right]. \quad (11·8)$$

The twisting couple M_3 is found from (10·34) to be

$$M_3 = \psi\frac{\pi ab}{2}(a^2+b^2)\left(\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)(\lambda^3-\alpha^2)$$
$$= \psi\frac{\pi a_0b_0}{2\lambda^2}(a_0^2+b_0^2)\left(\frac{C_1}{\lambda}+\frac{C_2}{\lambda^2}\right)(\lambda^3-\alpha^2).$$

When the elastic constant C_2 is zero, that is, when the material of which the cylinder is composed is the neo-Hookean incompressible material defined by Rivlin (1948*a*), and when there is no finite extension of the cylinder, the expression (11·8) becomes in this case

$$Y^3 = 6\psi^2\pi a_0b_0C_1\left\{h+\frac{3a_0^4-10a_0^2b_0^2+3b_0^4}{24(a_0^2+b_0^2)}\right\}.$$

We see that if there is to be zero total force over the ends of the cylinder we require

$$h = -\frac{(3a_0^2-b_0^2)(a_0^2-3b_0^2)}{24(a_0^2+b_0^2)},$$

and it is interesting to note that h will be positive or negative according as a_0/b_0 is less than or greater than $\surd 3$.

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